SPECTRA OF LARGE RANDOM MATRICES: A METHOD OF STUDY

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Abstract. A formalism for study of spectral correlations in non–Gaussian, unitary invariant ensembles of large random matrices with strong level confinement is reviewed. It is based on the Shohat method in the theory of orthogonal polynomials. The approach presented is equally suitable for description of both local and global spectral characteristics, thereby providing an overall look at the phenomenon of spectral universality in Random Matrix Theory.

1. Introduction: Motivation and Basic Results

1.1. UBIQUITY OF INVARIANT RANDOM MATRIX MODELS

Random matrices [1, 2] have been introduced in a physical context since the pioneering works by Wigner [3] and Dyson [4, 5]. Initially proposed as an effective phenomenological model for description of the higher excitations in nuclei [6] they found numerous applications in very diverse fields of physics such as two dimensional quantum gravity [7], quantum chromodynamics [8], quantum chaos [9], and mesoscopic physics [10, 11]. One can state that from the standpoint of the mathematical formalism all these fields are pooled by the Random Matrix Theory (RMT). Such an ubiquity of random matrices owes its origin to the exclusive role played by symmetry. The most amazing evidence to this fact comes from *invariant random*

matrix models which will be the focus of this review. The main feature of these models lies in that they discard (irrelevant) microscopic details of the physical system in question, but they do properly take into account its underlying fundamental symmetries. In accordance with the very idea of the construction of invariant matrix models, they do *not* relate to any dynamical properties of the physical object under study: general symmetry requirements alone lead to appearance of knowledge about the system. As far as the symmetry constraints follow from the first principles, the Random Matrix Theory turns out to be a general and powerful field–theoretical approach leading to a unified mathematical description of the quite different physical problems mentioned above.

A great variety of invariant random matrix ensembles can be assigned to three irreducible symmetry classes [5]. To specify them, we consider the typical line of arguing used in applications of the Random Matrix Theory to the disordered quantum mechanical systems, where it was first invented by Gor'kov and Eliashberg [12]. Since in this situation the microscopic Hamiltonian $\mathcal{H}$ is rather intricate, the integration of exact equations is impossible. It is therefore useful to appeal to statistical description by conjecturing that the operator $\mathcal{H}$ can be modelled by an $N \times N$ random matrix $\mathbf{H}$ whose eigenvalues and eigenvectors reproduce statistically the eigenlevels and eigenfunctions of the real microscopic Hamiltonian in the thermodynamic limit which corresponds to the matrix dimension N going to infinity. With this conjecture accepted, an ensemble of large random matrices $\mathbf{H}$, characterized by the joint distribution function $P[\mathbf{H}]$ of the matrix elements $\mathbf{H}_{ij}$ of the corresponding Hamiltonian $\mathcal{H}$, becomes the main object of study. Once the primary role of symmetry is postulated in the RMT–approach, the matrix $\mathbf{H}$ must adequately reflect the symmetry properties of the physical system under study. The matrix $\mathbf{H}$ is chosen to be real symmetric if the underlying physical system possesses time–reversal and rotational invariance. Systems with broken time–reversal symmetry are characterized by a Hermitean matrix $\mathbf{H}$, while systems with conserved time–reversal symmetry but with broken rotational invariance are described by a self–dual Hermitean matrix. These three symmetry classes referred to as the orthogonal, unitary and symplectic symmetry classes, respectively, can be characterized by a symmetry parameter β equals the number of independent elements in the off–diagonal entries of the matrix $\mathbf{H}$. The parameter $\beta = 1$ for a real symmetric matrix (orthogonal symmetry), $\beta = 2$ for a Hermitean matrix (unitary symmetry), and $\beta = 4$ for a self–dual Hermitean matrix (symplectic symmetry).

So far we did not yet specify the form of the joint distribution function $P[\mathbf{H}]$ of the matrix elements $\mathbf{H}_{ij}$. By definition, the invariant random

matrix ensembles are characterized by

$$P\left[\mathbf{H}\right]=\frac{1}{\mathcal{Z}_N}\exp\left\{-\beta\mathrm{Tr}V\left[\mathbf{H}\right]\right\}. \tag{1}$$

Here the function $V\left[\mathbf{H}\right]$ should ensure the existence of the partition function $\mathcal{Z}_N$, which is defined by the normalization condition $\int P\left[\mathbf{H}\right]d\left[\mathbf{H}\right]=1$ with the elementary volume $d\left[\mathbf{H}\right]$ depending on the symmetry of the matrix $\mathbf{H}$. For $\beta=1$ the volume element $d\left[\mathbf{H}\right]=\prod_{i\leq j}d\mathbf{H}_{ij}$, for $\beta=2$ the volume element $d\left[\mathbf{H}\right]=\prod_{i\leq j}d\,\mathrm{Re}\,\mathbf{H}_{ij}\prod_{i<j}d\mathrm{Im}\mathbf{H}_{ij}$, while for $\beta=4$ it equals $d\left[\mathbf{H}\right]=\prod_{i\leq j}d\mathbf{H}_{ij}^{(0)}\prod_{\sigma=1}^{3}\prod_{i<j}d\mathbf{H}_{ij}^{(\sigma)}$. The presence of the trace in Eq. (1) leads to the invariance of the probability density $P\left[\mathbf{H}\right]d\left[\mathbf{H}\right]$ under the similarity transformation $\mathbf{H}\rightarrow\mathcal{R}_\beta^{-1}\mathbf{H}\mathcal{R}_\beta$ with $\mathcal{R}_\beta$ being an orthogonal, unitary or symplectic $N\times N$ matrix for $\beta=1,2$ or 4, respectively. In turns, the invariance built in the probability density $P\left[\mathbf{H}\right]d\left[\mathbf{H}\right]$ implies that *there is no preferential basis in the space of matrix elements*. From the physical point of view, this means that invariant matrix models given by Eq. (1) are applicable to particular regimes of a physical system where (i) all the normalized linear combinations of the eigenstates have similar properties, and where (ii) the dimensionality is irrelevant. In disordered systems this is just a metallic state where the typical electron states are extended and hence structureless.

Notice that up to this point we have no constraints allowing us to uniquely choose the function $V\left[\mathbf{H}\right]$ (referred to as "confinement potential"). However, if we impose the additional requirement that the entries of the random matrix $\mathbf{H}$ be statistically independent of each other, we immediately arrive at the Gaussian Orthogonal, Unitary and Symplectic Ensembles (GOE, GUE, GSE) which are characterized by the quadratic confinement potential $V\left[\mathbf{H}\right]\propto\mathbf{H}^2$. This particular form of confinement potential leads to significant mathematical simplifications which allowed the complete treatment of these three ensembles many years ago [13].

It is remarkable that (even for non–Gaussian distributions of $P\left[\mathbf{H}\right]$) the invariant random matrix ensembles possess a great degree of mathematical tractability, and, what is more important, they have a high physical relevance, being much more than just a mathematical construction. In this respect we mention that in the physics of disordered systems the applicability of Gaussian invariant random matrix ensembles to description of weakly disordered systems has been proven by Efetov [14] by using the nonlinear σ–model. In that study [14] the statistical properties of energy levels for metallic particles with volume imperfections were considered by solving the Schrödinger equation with nonperturbative averaging over the random potential configurations within the framework of the supersymmetry method. Random Matrix Theory appears there as a zero–dimensional version of a

more general microscopic nonlinear σ–model, thereby proving the validity of the basic principles used for an RMT phenomenological description of energy levels of noninteracting electrons confined in a restricted volume. This connection takes place at times much larger than the ergodic time needed for diffusive particle to completely and homogeneously fill the available volume of the sample provided it is in the metallic regime characterized by a dimensionless Thouless conductance $g \gg 1$.

Let us point out that the choice of quadratic confinement potential $V[\mathbf{H}]$ can hardly be justified. Indeed, it was understood from the very beginning [13] that the requirement of statistical independence of the matrix elements $\mathbf{H}_{ij}$ is not motivated by the first principles and, therefore, the important problem of elucidating the influence of a particular form of confinement potential on the predictions of the Random Matrix Theory developed for Gaussian Invariant Ensembles had been posed already in the sixties. Despite this fact, considerable progress in study of spectral properties of *non–Gaussian* random matrix ensembles was achieved almost thirty years later when RMT experienced a great renaissance due to new ideas in the physics of disordered/chaotic systems which had led the birth of mesoscopic physics, as well as due to a penetration of Random Matrix Theory to quantum chromodynamics (QCD). In the latter field, the Random Matrix Theory turned out to be a useful tool for understanding the spectral properties of low–lying eigenvalues of the Dirac operator. The idea of introducing the RMT–approach in QCD is very similar to that in the physics of disordered systems and is based on the conjecture [8, 15] that the spectral density of the Dirac operator very close to the spectrum origin should depend only on the symmetries in question. One startling consequence of this conjecture is that the spectral density of the Dirac operator near the origin need not be computed within the framework of the gauge theories at all, but it can be extracted from much simpler random matrix ensembles reflecting the symmetry of the problem. The matrix models appearing in the context of QCD are manifestly non–Gaussian possessing an additional chiral structure[1] [16]. Another motivation for studying the spectral properties of non–Gaussian random matrix ensembles comes from the theories of $2D$ quantum gravity [7].

1.2. NON–GAUSSIAN RANDOM MATRIX ENSEMBLES AND

[1]Throughout the paper we consider non–chiral non–Gaussian random matrix ensembles. It can be shown that an arbitrary chiral matrix model can be reduced to an auxiliary one without chirality; see, for instance, Ref. [17].

The examples above clearly demonstrate an important role the non–Gaussian random matrix ensembles play in different physics theories, and serve as a compelling evidence of the necessity to have a powerful method for study of their spectral properties which could be equally applicable to rather different probability measures $P[\mathbf{H}]$. During (mostly) the recent decade a number of methods were developed in order to treat non–Gaussian random matrix ensembles. All of them can schematically be related to two groups.

1.2.1. *Global Universality*

The first group includes different approximate methods useful to explore *global spectral characteristics* which manifest themselves on the scale of $n \gg 1$ eigenlevels. Among these methods there are (i) the mean–field approximation proposed by Dyson [4, 18] which allows one to compute the density of levels in random matrix ensemble; (ii) the Schwinger–Dyson loop equations' technique [19, 20, 21] that had led to discovery of the phenomenon of the *global spectral universality*; (iii) the method of functional derivative of Beenakker [22, 23], and (iv) the diagrammatic approach of Brézin and Zee [24] whose development enabled their authors to study the phenomenon of global universality [19] in more detail as well as to generalize it in the context of mesoscopic physics.

It was found that contrary to the one–point spectral characteristics (such as one–point Green's function or level density) which essentially depend on the measure $P[\mathbf{H}]$, i.e. on the explicit form of the confinement potential, the *functional form* of (connected) two–point correlators becomes insensitive to the details of confinement potential upon smoothing over the scale which is much larger than the mean level spacing Δ_N but much smaller than the scale of the entire spectrum support. This is the essence of the phenomenon of global spectral universality. For example, the smoothed connected density–density correlator computed for the matrix model Eq. (1) is given by the universal function

$$\rho_c^{(N)}(\lambda,\lambda') = -\frac{1}{\pi^2\beta(\lambda-\lambda')^2}\frac{\mathcal{D}_N^2-\lambda\lambda'}{\left(\mathcal{D}_N^2-\lambda^2\right)^{1/2}\left(\mathcal{D}_N^2-\lambda'^2\right)^{1/2}}, \qquad (2)$$

where $\lambda \neq \lambda'$. This form of $\rho_c^{(N)}$ is valid for random matrices whose spectrum is supported on a single, symmetric interval $(-\mathcal{D}_N, +\mathcal{D}_N)$. Here, the universality implies that all the information about the particular form of $V[\mathbf{H}]$ is encoded into $\rho_c^{(N)}$ only through the end point $\mathcal{D}_N$ of the spectrum support.

The methods mentioned above, being applicable to study of the spectral correlations in the long–range regime for all three symmetry classes, leave

aside the fine structure of eigenvalue correlations manifested on the scale of the mean eigenvalue spacing. In this sense, these approaches are less informative as compared to the method of orthogonal polynomials [1] which, along with the supersymmetry approach [25, 26], enters the second group.

1.2.2. *Local Universality*

At present, the orthogonal polynomial technique, originally developed by Gaudin and Mehta [27], seems to be the most powerful one furnishing us the possibility to probe both the global spectral characteristics (which are of importance in computing the integral spectral properties) and the local spectral correlations (which describe a dynamics of underlying physical system) for arbitrary symmetry classes.

(i) The early attempts [28, 29, 30] to go beyond the Gaussian distribution of $P\left[\mathbf{H}\right]$ were concentrated on unitary invariant, $\mathrm{U}\left(N\right)$, matrix ensembles ($\beta = 2$) associated with classical orthogonal polynomials. It was found in Refs. [28, 29, 30, 31] that (in the thermodynamic limit $N \to \infty$) the scalar two–point kernel (in terms of which all n–point correlation functions are expressible, see Sec. 2.2) computed in the bulk of the spectrum, i.e. far from the end points of the eigenvalue support, follows the *sine law*

$$K_{\mathrm{bulk}}\left(s, s'\right) = \frac{\sin\left[\pi\left(s - s'\right)\right]}{\pi\left(s - s'\right)}, \tag{3}$$

that inevitably leads to the famous Wigner–Dyson level statistics [32] inherent in nuclear physics, weakly disordered and chaotic systems. This form of the scalar kernel corresponds to the *bulk scaling limit*, when the initial spectral variables λ, λ' are measured in the units of the mean eigenvalue spacing Δ_N, so that the scaled variable $s = \lambda/\Delta_N$. The same form of the scalar kernel Eq. (3) obtained for different ensembles associated with classical orthogonal polynomials was the first evidence to the phenomenon currently known as the *local spectral universality.* Later, it was shown in Ref. [31] that spectral properties of orthogonal $\mathrm{O}\left(N\right)$ and symplectic $\mathrm{Sp}\left(N\right)$ ($\beta = 1$ and $\beta = 4$) invariant ensembles associated with the weights of classical orthogonal polynomials[2] follow the predictions of GOE and GSE, respectively.

A further significant progress in the field came with the works [35, 36] whose authors considered spectral properties of $\mathrm{U}\left(N\right)$ invariant random matrix ensembles associated with strong symmetric confinement potentials of the form $V\left[\mathbf{H}\right] = \mathbf{H}^2 + \gamma\mathbf{H}^4$ ($\gamma > 0$) and $V\left[\mathbf{H}\right] = \sum_{k=1}^{p} a_k \mathbf{H}^{2k}$ ($a_p > 0$),

[2]In this situation the spectral correlations are expressible through the 2×2 matrix kernel, that can be computed by using so–called skew orthogonal polynomials [33]. In fact, as was recently shown in Ref. [34], the ensembles with $\beta = 1$ and $\beta = 4$ can be also treated without appealing to the skew polynomials.

respectively. Both works, based on different conjectures about the functional form of asymptotics of polynomials orthogonal with respect to a non–Gaussian measure, restricted their attention to the spectrum bulk, where the two–point kernel was shown, once again, to follow the sine law, Eq. (3). A rigorous treatment of a richer class of $\mathrm{U}(N)$ invariant random matrix ensembles related to the Freud and Erdös–type orthogonal polynomials was given in Refs. [37, 38]. It was proven there that the universal sine law for the two–point kernel holds for a wide class of monotonic (not necessarily of polynomial form) non–singular confinement potentials $V(\lambda)$ which increase at least as $|\lambda|$ at infinity, and can grow as or even faster than any polynomial at infinity. Confinement potentials satisfying these properties are referred to as *strong confinement potentials*. This definition takes its origin in the limits [37] of spectral universality and is non–accidently connected to the problem of determinate and indeterminate moments [39]. Also, it was demonstrated in Ref. [38] that an intimate connection exists between the structure of Szegö functional [40] entering the strong pointwise asymptotics of orthogonal polynomials and the mean–field equation by Dyson for mean level density that has been derived in Ref. [38] within the framework of orthogonal polynomial technique.

(ii) All the random matrix ensembles treated in Refs. [28, 29, 30, 31, 35, 36, 37, 38] were characterized by strong confinement potential with no singularities. However, (logarithmic) singularities do appear when one considers chiral matrix ensembles, arising in the context of QCD [8], in the theory of mesoscopic electron transport [41] and in description of electron level statistics in normalconducting–superconducting hybrid structures [42]. An example of a rather general (though non–chiral) random matrix ensemble possessing a log–singular level confinement is given by the distribution

$$P[\mathbf{H}] = \frac{1}{\mathcal{Z}_N} |\det \mathbf{H}|^{\alpha\beta} \exp\left\{-\beta \mathrm{Tr} V[\mathbf{H}]\right\}, \tag{4}$$

where the function $V[\mathbf{H}]$ is a well behaved function which has not singular points. Ensemble Eq. (4), being a natural generalization of the matrix ensemble proposed by Bronk [30], was first considered in the situations where associated orthogonal polynomials were classical [30, 43, 44, 45]. For quadratic confinement potential and $\beta = 2$ one obtains that in the vicinity of the singularity, $\lambda = 0$, the scalar kernel satisfies the *Bessel law*

$$K_{\mathrm{orig}}(s, s') = \frac{\pi}{2} (ss')^{1/2} \frac{J_{\alpha+1/2}(\pi s) J_{\alpha-1/2}(\pi s') - J_{\alpha-1/2}(\pi s) J_{\alpha+1/2}(\pi s')}{s - s'}, \tag{5}$$

where s and s' are scaled by the level spacing $\Delta_N(0)$ near the spectrum origin, $s = \lambda/\Delta_N(0)$, and $\alpha > -1/2$. This scaling procedure is referred to as the *origin scaling limit.* Extensions to two other symmetry classes, as well

as to the chiral matrix ensembles, can be found in Refs. [43, 44, 45, 46]. An important breakthrough in understanding the universal character of this kernel was given in Refs. [47, 17]. These authors, guided by QCD applications, have shown that the Bessel kernel Eq. (5) is again universal, being independent of the details of strong confinement potential V [**H**]. An alternative proof of universality that holds more generally was presented in Ref. [48].

(iii) The third type of universal correlations takes place near the soft edge of the spectrum support, which is of special interest in the models of two–dimensional quantum gravity. The first study of the level density near the end point of the spectrum support is due to Wigner [49]. More comprehensive description of the tails of the density of states was done in Ref. [50]. It was shown there that at $\beta = 2$ a universal crossover occurs from a nonzero density of states to a vanishing one that is independent of the confining potential in the *soft–edge scaling limit.* Later it was demonstrated [51] that eigenvalue *correlations* in the $\mathrm{U}(N)$ invariant matrix ensembles with quartic and sextic confinement potentials are determined by the scalar kernel obeying the *Airy law* [52]

$$K_{\mathrm{soft}}(s, s') = \frac{\mathrm{Ai}(s)\,\mathrm{Ai}'(s') - \mathrm{Ai}(s')\,\mathrm{Ai}'(s)}{s - s'}, \tag{6}$$

suggesting that the Airy kernel should be universal as well. (Here the rescaling $s \propto N^{2/3}(\lambda/\mathcal{D}_N - 1)$ determines the soft–edge scaling limit). This conjecture has been proven in Ref. [53], where it was also shown that the Airy correlations, being universal for a class of matrix models with monotonic confinement potential or with that having light local extrema, are indeed a particular case of more general universal multicritical correlations [53].

1.2.3. *Universality in a Broader Context*

Are the universal scalar kernels in unitary invariant random matrix models with strong level confinement exhausted by the universal sine, Bessel and Airy laws given by Eqs. (3), (5) and (6) above? The present state of the art leads us to the negative answer. Indeed, by adding a singular component to the strong level confinement we may either accumulate a finite number of eigenvalues near the singular point λ_{sing} or repel them from it. Such a rearrangement of eigenlevels will lead to emergence of a new scalar two–point kernel in the vicinity of the singular point of the spectrum. (In particular case of the logarithmic singularity this kernel will follow the Bessel law Eq. (5)). Generically, it is natural to expect that the functional form of the kernel near λ_{sing} will be sensitive to the particular type of the singular deformation. However, the new two–point kernel will be insensitive to the details of the background component of the confinement potential, as

it takes place in the case of the Bessel kernel. In this sense, one can still say about a phenomenon of universality. One of the latest evidences to this fact can be found in Ref. [54] where deformed ensembles of large random matrices associated with massive Dirac operators were considered.

By the same token, the universal global spectral correlator expressed by Eq. (2) is not the only one arising in Random Matrix Theory. As it was already stressed, the universal function Eq. (2) is inherent in matrix models with spectra possessing a single connected support. Correspondingly, ensembles of large random matrices with eigenvalue densities having more than one–cut support will give rise to the novel global spectral correlators whose universality classes (for a given symmetry parameter β) are characterized entirely by the number of cuts in the support of spectral density. This was explicitly demonstrated in the recent studies [20, 55] by means of the loop equation technique.

1.3. THE AIM

At this point, it is appropriate to notice that all the mentioned above (universal) results for both global and local eigenvalue correlations have been obtained by using different methods, each of them was only suitable for a particular problem under consideration. Any deformations of $P\left[\mathbf{H}\right]$ (which will preserve its invariance) would cause principal difficulties in elucidating the influence of these deformations on spectral properties of corresponding random matrix ensembles. The goal of this paper is to represent a general method (recently developed in a series of publications [53, 48, 56]), which is equally suitable for study of both local and global eigenvalue correlations, and easily leads to generalizations. The approach we introduce is based on a simple and elegant idea [57] by J. Shohat (which goes back to 1930), providing a detailed description of the spectral properties of non–Gaussian $\mathrm{U}\left(N\right)$ invariant random matrix ensembles through the analysis of the three–term recurrence equation for associated orthogonal polynomials. We show that for the most situations of interest, the knowledge of the large–N behavior of the coefficients in the recurrence equation is sufficient to directly reconstruct the local eigenvalue correlations of arbitrary order, as well as to explore the global spectral statistics. In the case of a non–singular, well behaved confinement potential, the knowledge of such a large–N behavior of the recurrence coefficients is equivalent, in fact, to a knowledge of the Dyson density of states for the corresponding random matrix ensemble. The latter assertion leads to a rather unexpected conclusion: Once the Dyson density of states (which is a rather crude one–point spectral characteristics) is available, the scalar kernel (and hence the n–point spectral correlators) can immediately be recovered through the solution of a certain

second–order differential equation (See Sec. 4.3).

We believe that this method offers not only new computational potentialities, but also provides a different, overall look at the problem of eigenvalue correlations in unitary invariant random matrix ensembles in arbitrary spectrum range and in arbitrary scaling limits. It seems that together with the very recent works [58, 59] establishing a precise connection of the scalar kernel for random matrix ensembles with $\mathrm{U}(N)$ symmetry with the 2×2 matrix kernels in ensembles with $\mathrm{O}(N)$ and $\mathrm{Sp}(N)$ symmetries, the formalism to be reviewed below gives a rather complete solution of the problem of eigenvalue correlations in invariant matrix models with strong level confinement.

The review is organized as follows. Section 2 contains a brief description of the Gaudin–Mehta calculational scheme, that introduces the orthogonal polynomials as a tool for exact evaluation of n–point correlation functions in $\mathrm{U}(N)$ invariant random matrix ensembles. In Section 3 the Shohat method in the theory of orthogonal polynomials is presented. Section 4 is devoted to a detailed study of spectral properties of large Hermitean random matrices with a single connected eigenvalue support. In Section 5 we extend this analysis to random matrices with eigenvalue gap. Section 6 contains conclusions. The most lengthy calculations are collected in three Appendices.

2. Elements of the Gaudin–Mehta formalism

2.1. INVARIANT RANDOM MATRIX MODEL IN EIGENVALUE REPRESENTATION: TWO INTERPRETATIONS

The invariance of the distribution function $P[\mathbf{H}]$ implies that different matrices with the same eigenvalues have the same probability of occurring. To study spectral characteristics of an invariant random matrix model it is convenient to integrate out "auxiliary" angular variables in the construction $P[\mathbf{H}]\, d[\mathbf{H}]$ in order to get the matrix model in the eigenvalue representation. To proceed with this, we have to pass from the integration over independent elements $\mathbf{H}_{ij}$ of the matrix $\mathbf{H}$ to the integration over the smaller space of its N eigenvalues $\{\lambda\}$, calculating the corresponding Jacobian J.

Let us introduce the matrix $\mathcal{R}_\beta$ that diagonalizes the random matrix $\mathbf{H}$, so that $\mathbf{H} = \mathcal{R}_\beta^{-1} \Lambda \mathcal{R}_\beta$ and $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N)$, and consider the infinitesimal variation of $\mathbf{H}$,

$$\delta\mathbf{H} = \mathcal{R}_\beta^{-1} \left(\mathcal{R}_\beta \delta \mathcal{R}_\beta^{-1} \Lambda + \delta\Lambda + \Lambda \delta \mathcal{R}_\beta \mathcal{R}_\beta^{-1} \right) \mathcal{R}_\beta = \mathcal{R}_\beta^{-1} \left(\delta\Lambda - i\left[\Lambda, \delta s\right] \right) \mathcal{R}_\beta. \tag{7}$$

Here we have denoted $\delta s = i \delta \mathcal{R}_\beta \mathcal{R}_\beta^{-1}$. The norm, $\|\delta\mathbf{H}\|^2 = \mathrm{Tr}\,(\delta\mathbf{H})^2$, of the infinitesimal variation of $\mathbf{H}$ is

$$\|\delta\mathbf{H}\|^2 = \mathrm{Tr}\,(\delta\Lambda)^2 - 2i\,\mathrm{Tr}\left(\left[\delta\Lambda, \Lambda\right] \delta s\right) + 2\,\mathrm{Tr}\left(-\delta s \Lambda \delta s \Lambda + (\delta s)^2 \Lambda^2 \right). \tag{8}$$

The second term in the last expression vanishes as the matrices $\delta\Lambda$ and Λ are diagonal. We then find

$$\|\delta\mathbf{H}\|^2 = \sum_i (\delta\lambda_i)^2 + \sum_{i,j} (\lambda_i - \lambda_j)^2 |\delta s_{ij}|^2 . \tag{9}$$

The independent variables are the variations of the eigenvalues $\delta\lambda_i$ and δs_{ij} for $\beta = 1$, $\mathrm{Re}\,\delta s_{ij}$, $\mathrm{Im}\,\delta s_{ij}$ for $\beta = 2$, or $\delta s_{ij}^{(\sigma)}$ with $\sigma = 0, 1, 2, 3$ for $\beta = 4$, $i < j$. Then, from Eq. (9) we obtain the Jacobian J of the transformation $\mathbf{H} \mapsto \{\lambda_i, \mathcal{R}_\beta\}$ that is equal to $\sqrt{\det G}$, where G is the metric tensor with $\det G = \prod_{i \neq j} |\lambda_i - \lambda_j|^\beta$. Hence

$$J = \prod_{i<j} |\lambda_i - \lambda_j|^\beta = |\Delta(\lambda)|^\beta \tag{10}$$

with $\Delta(\lambda)$ being the Vandermonde determinant. The construction Eq. (10) is also known as the Jastrow factor.

Combining Eqs. (1) and (10) we arrive at the famous expression for the joint probability density function of the eigenvalues $\{\lambda\}$ of the matrix $\mathbf{H}$:

$$\rho(\lambda_1, \ldots, \lambda_N) = \mathcal{Z}_N^{-1} \exp\{-\beta \sum_{i=1}^{N} V(\lambda_i)\} |\Delta(\lambda)|^\beta \tag{11}$$

$$= \mathcal{Z}_N^{-1} \exp\{-\beta[\sum_i V(\lambda_i) - \sum_{i<j} \ln|\lambda_i - \lambda_j|]\}. \tag{12}$$

The probability distribution given by Eq. (12) has the form of a Gibbs distribution for a classical one–dimensional system of N "particles" at "positions" λ_i confined by the external one–body potential $V(\lambda)$ and *interacting* with each other through the pairwise logarithmic law originated from the Jacobian J of the transformation $\mathbf{H} \mapsto \{\lambda_i, \mathcal{R}_\beta\}$. The symmetry parameter β plays the role of the equilibrium temperature. Such an interpretation of Eq. (12) gives rise to approximate mean–field methods in the Random Matrix Theory.

For invariant matrix ensembles with unitary symmetry ($\beta = 2$), the probability distribution in the form of Eq. (11) can alternatively be related to a system of fictitious *non–interacting fermions*, that can be described by effective one–particle Schrödinger equation [53]. This equation is a cornerstone of the method under review. Although such a simple, transparent interpretation cannot be ascribed to $\rho(\lambda_1, ..., \lambda_N)$ for two other symmetry classes, $\beta = 1$ and $\beta = 4$, a recently discovered deep connection [58, 59] between orthogonal, unitary and symplectic ensembles makes the unitary invariant ensembles of random matrices to be the central and most important.

2.2. ORTHOGONAL POLYNOMIALS' TECHNIQUE: $\beta = 2$

For $\beta = 2$ the structure of Eq. (11) enables us to exactly represent all the statistical characteristics of the spectrum in the terms of polynomials orthogonal with respect to a non–Gaussian measure $d\alpha$. To demonstrate this, it is useful to write down the joint probability density function of the eigenvalues $\{\lambda\}$ in the form

$$\rho(\lambda_1, \ldots, \lambda_N) = \Psi_0^2(\lambda_1, \ldots, \lambda_N), \tag{13}$$

$$\Psi_0(\lambda_1, \ldots, \lambda_N) = \mathcal{Z}_N^{-1/2} \exp\{-\sum_{i=1}^{N} V(\lambda_i)\} \Delta(\lambda). \tag{14}$$

One can see that Ψ_0 can be thought of as a wave function of N fictitious non–interacting fermions. Namely, noting that

$$\Delta(\lambda) = \prod_{i>j=1}^{N} (\lambda_i - \lambda_j) = \det \left\| \lambda_i^{j-1} \right\|, \tag{15}$$

and taking the linear combinations of the columns of the initial matrix with entries λ_i^{j-1}, one can reduce this matrix to the matrix whose entries are arbitrary polynomials $P_{j-1}(\lambda_i)$ of degrees $j - 1 = 0, 1, \ldots, N - 1$,

$$\Delta(\lambda) = \det \| P_{j-1}(\lambda_i) \|. \tag{16}$$

Further, choosing these polynomials to be orthogonal with respect to the measure $d\alpha(\lambda) = \exp\{-2V(\lambda)\} d\lambda$,

$$\int_{\lambda \in (-\infty, +\infty)} d\alpha(\lambda) P_n(\lambda) P_m(\lambda) = \delta_{nm}, \tag{17}$$

we arrive at the conclusion that the function Ψ_0 in Eq. (14) can be represented as a Slater determinant

$$\Psi_0(\lambda_1, \ldots, \lambda_N) = \frac{1}{\sqrt{N!}} \det \| \varphi_{j-1}(\lambda_i) \| \tag{18}$$

that formally describes the system of N fictitious non–interacting fermions located at "spatial" points λ_i and characterized by the set of orthonormal "eigenfunctions"

$$\varphi_n(\lambda) = P_n(\lambda) \exp\{-V(\lambda)\}, \tag{19}$$

$$\int_{-\infty}^{+\infty} d\lambda \varphi_n(\lambda) \varphi_m(\lambda) = \delta_{nm}. \tag{20}$$

Bearing in mind the representation Eq. (13), and taking into account Eqs. (17) – (19), we readily get that

$$\rho\left(\lambda_1,\ldots,\lambda_N\right)=\frac{1}{N!}\det\left\|K_N\left(\lambda_i,\lambda_j\right)\right\|_{i,j=1\ldots N}, \tag{21}$$

where

$$K_N\left(\lambda,\lambda'\right)=\sum_{k=0}^{N-1}\varphi_k\left(\lambda\right)\varphi_k\left(\lambda'\right) \tag{22}$$

is the *scalar two–point kernel*. Making use of the Christoffel–Darboux theorem [60] (see Eq. (32) below), the two–point kernel can be reduced to the form

$$K_N\left(\lambda,\lambda'\right)=c_N\frac{\varphi_N\left(\lambda'\right)\varphi_{N-1}\left(\lambda\right)-\varphi_N\left(\lambda\right)\varphi_{N-1}\left(\lambda'\right)}{\lambda'-\lambda}, \tag{23}$$

convenient for the further analysis. In Eq. (23) c_N is the coefficient in the three–term recurrence equation (see Eq. (28) below) for the set P_n of the polynomials orthogonal with respect to the measure $d\alpha$. This formula simplifies significantly RMT calculations, since the "eigenfunctions" φ_N with large "quantum numbers" $N\gg 1$ entering Eq. (23) can be replaced by their large–N asymptotics.

The n–point correlation function is determined in the Random Matrix Theory by the formula [1]

$$R_n\left(\lambda_1,\ldots,\lambda_n\right)=\frac{N!}{(N-n)!}\prod_{k=n+1}^{N}\int_{-\infty}^{+\infty}d\lambda_k\rho\left(\lambda_1,\ldots,\lambda_N\right). \tag{24}$$

It describes the probability to find n levels around each of the points $\lambda_1,\ldots,\lambda_n$ when the positions of the remaining levels are unobserved. The multiple integrals in the last equation can exactly be calculated by using the representation Eq. (21). The result of the integration reads [1]

$$R_n\left(\lambda_1,\ldots,\lambda_n\right)=\det\left\|K_N\left(\lambda_i,\lambda_j\right)\right\|_{i,j=1\ldots n}. \tag{25}$$

Equation (25) implies that the knowledge of the scalar two–point kernel $K_N\left(\lambda,\lambda'\right)$ is sufficient to calculate the energy level correlation function of any order. In particular, the averaged density of states is expressed as

$$\left\langle\nu_N\left(\lambda\right)\right\rangle=R_1\left(\lambda\right)=K_N\left(\lambda,\lambda\right). \tag{26}$$

Analogously, Eq. (25) leads to the following expression for connected "density-density" correlation function $\rho_c^{(N)}=\left\langle\nu_N\left(\lambda\right)\nu_N\left(\lambda'\right)\right\rangle-\left\langle\nu_N\left(\lambda\right)\right\rangle\left\langle\nu_N\left(\lambda'\right)\right\rangle$,

$$\begin{aligned}\rho_c^{(N)} &= \delta\left(\lambda-\lambda'\right)R_1\left(\lambda\right)+R_2\left(\lambda,\lambda'\right)\\ &= \delta\left(\lambda-\lambda'\right)K_N\left(\lambda,\lambda\right)-K_N^2\left(\lambda,\lambda'\right).\end{aligned} \tag{27}$$

Thus, all the nontrivial information about eigenlevel correlations is contained in the squared two–point kernel $-K_N^2(\lambda, \lambda')$.

3. The Shohat Method: General Relations

Equations (13) and (18) suggest that the joint distribution function of N eigenvalues of a $\mathrm{U}(N)$ invariant random matrix ensemble can be interpreted as a probability of finding N fictitious non–interacting fermions to be confined in a one–dimensional space. The effective one–particle Schrödinger equation for the wave functions $\varphi_n(\lambda)$, Eq. (19), of these fictitious fermions can be derived by mapping a three–term recurrence equation for orthogonal polynomials, Eq. (17), onto a second–order differential equation. The method of reducing a recurrence equation to a differential equation is essentially due to J. Shohat who proved in 1939 that orthogonal polynomials associated with exponential weights satisfy a second–order differential equation [57]. Much later the Shohat method has got a further development in the work by Bonan and Clark [61]. By now, rather extended mathematical literature exists on this subject [62, 63, 64].

Let as consider a set of polynomials $P_n(\lambda)$ orthogonal on the entire real axis with respect to the measure $d\alpha(\lambda) = \exp\{-2V(\lambda)\}\, d\lambda$. If $V(\lambda)$ is an even function[3], $V(-\lambda) = V(\lambda)$, this set of orthogonal polynomials can be defined by the recurrence equation

$$\lambda P_{n-1}(\lambda) = c_n P_n(\lambda) + c_{n-1} P_{n-2}(\lambda), \tag{28}$$

where the coefficients c_n are uniquely determined by the measure $d\alpha$.

In order to derive the differential equation for the wave functions $\varphi_n(\lambda) = P_n(\lambda)\exp\{-V(\lambda)\}$, we note that the following identity takes place,

$$\frac{dP_n(\lambda)}{d\lambda} = A_n(\lambda) P_{n-1}(\lambda) - B_n(\lambda) P_n(\lambda), \tag{29}$$

with functions $A_n(\lambda)$ and $B_n(\lambda)$ to be determined from the following consideration. Since $dP_n(\lambda)/d\lambda$ is a polynomial of the degree $n-1$, it can be represented [60] through the Fourier expansion in the terms of the kernel $Q_n(t,\lambda) = \sum_{k=0}^{n-1} P_k(\lambda) P_k(t)$ as

$$\frac{dP_n(\lambda)}{d\lambda} = \int d\alpha(t) \frac{dP_n(t)}{dt} Q_n(t,\lambda). \tag{30}$$

Integrating by parts in the last equation we get that

$$\frac{dP_n(\lambda)}{d\lambda} = 2\int d\alpha(t)\, Q_n(t,\lambda)\left(\frac{dV}{dt} - \frac{dV}{d\lambda}\right) P_n(t). \tag{31}$$

[3]For asymmetric confinement potentials the recurrence equation takes the form $(\lambda - b_n) P_{n-1}(\lambda) = c_n P_n(\lambda) + c_{n-1} P_{n-2}(\lambda)$. The additional parameter b_n can easily be incorporated into the calculational scheme.

Now, making use of the Christoffel–Darboux identity [60]

$$Q_n(t,\lambda) = \sum_{k=0}^{n-1} P_k(\lambda) P_k(t) = c_n \frac{P_n(t) P_{n-1}(\lambda) - P_n(\lambda) P_{n-1}(t)}{t-\lambda}, \quad (32)$$

we are led to

$$\begin{aligned} \frac{dP_n(\lambda)}{d\lambda} &= 2c_n \int d\alpha(t) \frac{V'(t) - V'(\lambda)}{t-\lambda} P_n(t) \\ &\quad \times [P_n(t) P_{n-1}(\lambda) - P_n(\lambda) P_{n-1}(t)]. \end{aligned} \quad (33)$$

Comparison of this expression with Eq. (29) yields

$$A_n(\lambda) = 2c_n \int d\alpha(t) \frac{V'(t) - V'(\lambda)}{t-\lambda} P_n^2(t), \quad (34)$$

$$B_n(\lambda) = 2c_n \int d\alpha(t) \frac{V'(t) - V'(\lambda)}{t-\lambda} P_n(t) P_{n-1}(t). \quad (35)$$

Now one can obtain the *exact* differential equation for the eigenfunctions φ_n. Differentiating Eq. (29), consequently applying Eqs. (29) and (28), and taking into account Eq. (19) we derive after somewhat lengthy calculations

$$\frac{d^2\varphi_n(\lambda)}{d\lambda^2} - \mathcal{F}_n(\lambda) \frac{d\varphi_n(\lambda)}{d\lambda} + \mathcal{G}_n(\lambda) \varphi_n(\lambda) = 0, \quad (36)$$

where

$$\mathcal{F}_n(\lambda) = \frac{1}{A_n} \frac{dA_n}{d\lambda}, \quad (37)$$

$$\begin{aligned} \mathcal{G}_n(\lambda) &= \frac{dB_n}{d\lambda} + \frac{c_n}{c_{n-1}} A_n A_{n-1} - B_n \left(B_n + 2\frac{dV}{d\lambda} + \frac{1}{A_n} \frac{dA_n}{d\lambda} \right) \\ &\quad + \frac{d^2V}{d\lambda^2} - \left(\frac{dV}{d\lambda} \right)^2 - \frac{1}{A_n} \frac{dA_n}{d\lambda} \frac{dV}{d\lambda}. \end{aligned} \quad (38)$$

When deriving Eqs. (36), (37) and (38) we made use of the sum rule

$$B_n + B_{n-1} - \frac{\lambda}{c_{n-1}} A_{n-1} = -2\frac{dV}{d\lambda}, \quad (39)$$

directly following from Eqs. (34), (35), (28) and from oddness of $dV/d\lambda$.

Equation (36) is valid for arbitrary n. Previously, an equation of this type was known in the context of the Random Matrix Theory only for GUE, where $V(\lambda) = \lambda^2/2$. For such a confinement potential both functions $A_n(\lambda)$ and $B_n(\lambda)$ can readily be obtained from Eqs. (34) and (35), and are given by $A_n(\lambda) = 2c_n$ and $B_n(\lambda) = 0$. Taking into account that for GUE

$c_n = (n/2)^{1/2}$, we end up with $\mathcal{F}_n(\lambda) = 0$ and $\mathcal{G}_n(\lambda) = 2n + 1 - \lambda^2$. This allows us to interpret $\varphi_n(\lambda)$ as a wave function of the fermion confined by a parabolic potential,

$$\frac{d^2\varphi_n^{\mathrm{GUE}}(\lambda)}{d\lambda^2} + \left(2n + 1 - \lambda^2\right)\varphi_n^{\mathrm{GUE}}(\lambda) = 0. \tag{40}$$

The effective Schrödinger equation (36) applies to general non–Gaussian random matrix ensembles as well, although the explicit calculation of $\mathcal{F}_n(\lambda)$ and $\mathcal{G}_n(\lambda)$ in this situation is a rather complicated task. In two cases of relatively simple measures with $V(\lambda) = \lambda^4/8 + q_3\lambda^3/6 + q_2\lambda^2/4 + q_1\lambda/2$ and $V(\lambda) = \lambda^6/12$ the functions $\mathcal{F}_n(\lambda)$ and $\mathcal{G}_n(\lambda)$ are known explicitly [62]. Significant simplifications, however, arise in the limit $n = N \gg 1$, which is just a thermodynamic limit of the Random Matrix Theory representing for us the most interest.

4. Random Matrices with Single Eigenvalue Support

In this Section we will be interested in the study of eigenvalue statistics for unitary invariant non–Gaussian large random matrices characterized by a distribution function $P[\mathbf{H}]$ given by Eq. (4). The confinement potential associated with this model is

$$V_\alpha(\lambda) = v(\lambda) - \alpha \log|\lambda|. \tag{41}$$

Here $v(\lambda)$ is the regular part of level confinement

$$v(\lambda) = \sum_{k=1}^{p} \frac{d_k}{2k}\lambda^{2k}, \tag{42}$$

with a positive leading coefficient, $d_p > 0$; the signs of the rest of the d_k can be arbitrary but they should lead to an eigenvalue density supported on a single connected interval, $\{\lambda\} \in (-\mathcal{D}_N, +\mathcal{D}_N)$. The parameter $\alpha > -1/2$ is the strength of the logarithmic singularity.

In Subsection 4.1 below, we demonstrate how the one–point spectral characteristics (density of states) can be obtained by making use of the recurrence equation (28). In Subsection 4.2, we turn to the study of the smoothed connected "density–density" correlator, also starting with recurrence equation (28). Finally, in Subsection 4.3, we obtain the universal scalar kernels in the origin, bulk and soft–edge scaling limits by solving the effective Schrödinger equation for fictitious fermions.

4.1. MACROSCOPIC LEVEL DENSITY FROM RECURRENCE EQUATION

We start with an explanation of the main idea of the derivation to make clear all the subsequent calculations. Here we mainly follow Ref. [65]. Our

basic observation is that in the large–N limit the density of states[4] $\nu_N(\lambda)$ consists of two parts,

$$\nu_N(\lambda) = \nu_N^{\rm smooth}(\lambda) + \nu_N^{\rm osc}(\lambda). \tag{43}$$

The smooth part $\nu_N^{\rm smooth}(\lambda)$ contributes to different integral characteristics determined by the density of states, while the oscillating part does not, because any integration will level the oscillating features. Then, for some smooth, well behaved, even function $f(\lambda)$ we have

$$\int_{-\infty}^{+\infty} d\lambda f(\lambda)\nu_N(\lambda) \stackrel{N\to\infty}{\to} \int_{\lambda\in{\rm support}} d\lambda f(\lambda)\nu_N^{\rm smooth}(\lambda). \tag{44}$$

Let us implement this scheme by choosing (without any loss of generality) $f(\lambda) = \lambda^{2s}$, with s being a positive integer. This choice is possible due to the evenness of $\nu_N(\lambda)$. By definition Eq. (26), we obtain from Eq. (23)

$$\nu_N(\lambda) = c_N \exp\{-2V_\alpha(\lambda)\}\left(P_N^{(\alpha)\prime}(\lambda)P_{N-1}^{(\alpha)}(\lambda) - P_{N-1}^{(\alpha)\prime}(\lambda)P_N^{(\alpha)}(\lambda)\right). \tag{45}$$

Having in mind the relation Eq. (29), the sum rule Eq. (39) and the definition Eq. (19) we come down to

$$\begin{aligned}\nu_N(\lambda) &= c_N\left[A_N^{(\alpha)}(\lambda)\left(\varphi_{N-1}^{(\alpha)}(\lambda)\right)^2 + \frac{c_N}{c_{N-1}}A_{N-1}^{(\alpha)}(\lambda)\left(\varphi_N^{(\alpha)}(\lambda)\right)^2\right. \\ &\left. -\varphi_N^{(\alpha)}(\lambda)\varphi_{N-1}^{(\alpha)}(\lambda)\left(\frac{\lambda}{c_{N-1}}A_{N-1}^{(\alpha)}(\lambda) + B_N^{(\alpha)}(\lambda) - B_{N-1}^{(\alpha)}(\lambda)\right)\right].\end{aligned} \tag{46}$$

Here the upper index (α) is used to reflect the presence of a log–singular component in confinement potential $V_\alpha(\lambda)$, Eq. (41). Thus, the level density is expressed through the wave functions $\varphi_N^{(\alpha)}(\lambda)$, and through the functions $A_N^{(\alpha)}(\lambda)$ and $B_N^{(\alpha)}(\lambda)$, given by Eqs. (34) and (35).

For further convenience we introduce two quantities,

$$\Lambda_{2\sigma}^{(N)} = \int d\alpha(t)\left(P_N^{(\alpha)}(t)\right)^2 t^{2\sigma} = \int_{-\infty}^{+\infty} dt\left(\varphi_N^{(\alpha)}(t)\right)^2 t^{2\sigma}, \tag{47}$$

$$\Gamma_{2\sigma+1}^{(N)} = \int d\alpha(t)P_N^{(\alpha)}(t)P_{N-1}^{(\alpha)}(t)t^{2\sigma+1} = \int_{-\infty}^{+\infty} dt\varphi_N^{(\alpha)}(t)\varphi_{N-1}^{(\alpha)}(t)t^{2\sigma+1}, \tag{48}$$

for which the alternative explicit integral representations are shown to exist in the Appendix A. Without going into details of those calculations, we

[4]Hereafter we use the notation $\nu_N(\lambda)$ for averaged density of states, Eq. (26).

only stress an extremely important role played by both the large–N limit, Eq. (124), of the recurrence equation for associated orthogonal polynomials and the asymptotic expansion, Eq. (125), deduced from Eq. (124). We also notice that owing to the existence of these explicit representations, derived by using a large–N version of the recurrence equation (28), all further calculations became possible.

The functions $A_N^{(\alpha)}$ and $B_N^{(\alpha)}$ entering Eq. (46) can be expressed in terms of $\Lambda_\sigma^{(N)}$ and $\Gamma_\sigma^{(N)}$. Namely, bearing in mind the definitions given by Eqs. (34) and (35) we obtain for $A_N^{(\alpha)}(\lambda) = A_{\rm reg}^{(N)}(\lambda) + \alpha A_{\rm sing}^{(N)}(\lambda)$,

$$A_{\rm reg}^{(N)}(\lambda) = 2c_N \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Lambda_{2(k-\sigma)}^{(N)} \lambda^{2\sigma-2}, \tag{49}$$

$$A_{\rm sing}^{(N)}(\lambda) = 2c_N \int \frac{d\alpha(t)}{t} \left(P_N^{(\alpha)}(t)\right)^2. \tag{50}$$

In the same way, the function $B_N^{(\alpha)}(\lambda) = B_{\rm reg}^{(N)}(\lambda) + \alpha B_{\rm sing}^{(N)}(\lambda)$ is given by

$$B_{\rm reg}^{(N)}(\lambda) = 2c_N \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Gamma_{2(k-\sigma)-1}^{(N)} \lambda^{2\sigma-1}, \tag{51}$$

$$B_{\rm sing}^{(N)}(\lambda) = \frac{2c_N}{\lambda} \int \frac{d\alpha(t)}{t} P_N^{(\alpha)}(t) P_{N-1}^{(\alpha)}(t). \tag{52}$$

When deriving these formulas we have used the fact of evenness of the measure $d\alpha(t)/dt$ and of $\left(P_N^{(\alpha)}(t)\right)^2$, as well as the expansion

$$\frac{t^k - \lambda^k}{t - \lambda} = \sum_{m=1}^{k} t^{k-m} \lambda^{m-1}. \tag{53}$$

The "singular" components, $A_{\rm sing}^{(N)}$ and $B_{\rm sing}^{(N)}$, can easily be determined. First, due to oddness of the integrand in Eq. (50), we have $A_{\rm sing}^{(N)}(\lambda) \equiv 0$. Second, in order to find $B_{\rm sing}^{(N)}$, we notice that the quantity

$$\gamma_n^{(\alpha)} = c_n \int \frac{d\alpha(t)}{t} P_n^{(\alpha)}(t) P_{n-1}^{(\alpha)}(t), \tag{54}$$

where n is not necessarily large, obeys the identity $\gamma_n^{(\alpha)} + \gamma_{n-1}^{(\alpha)} = 1$, which is a direct consequence of the recurrence equation (28). As far as $\gamma_{2n}^{(\alpha)} =$

$\gamma_{2n-2}^{(\alpha)} = \ldots = \gamma_2^{(\alpha)} \equiv 0$, we conclude that $\gamma_n^{(\alpha)} = \left[1-(-1)^n\right]/2$ and therefore, $B_{\rm sing}^{(N)}(\lambda) = 2\gamma_N^{(\alpha)}/\lambda$. Hence we are led to the following representations,

$$A_N^{(\alpha)}(\lambda) = 2c_N \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Lambda_{2(k-\sigma)}^{(N)} \lambda^{2\sigma-2}, \tag{55}$$

$$B_N^{(\alpha)}(\lambda) = 2c_N \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Gamma_{2(k-\sigma)-1}^{(N)} \lambda^{2\sigma-1} + \alpha \frac{1-(-1)^N}{\lambda}. \tag{56}$$

With these preliminarily calculations in hand we are ready to implement the idea of recovering the Dyson density of states from the recurrence equation. In accordance with Eqs. (44) and (46) there are five contributions to the integral in the r.h.s. of Eq. (44) corresponding to five terms in Eq. (46). Substituting Eqs. (55) and (56) into Eq. (46) and performing a formal integration with the help of Eqs. (47) and (48), these contributions are found to be

$$\rho_1 = 2c_N^2 \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Lambda_{2(k-\sigma)}^{(N)} \Lambda_{2(\sigma+s-1)}^{(N-1)}, \tag{57}$$

$$\rho_2 = 2c_N^2 \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Lambda_{2(k-\sigma)}^{(N-1)} \Lambda_{2(\sigma+s-1)}^{(N)}, \tag{58}$$

$$\rho_3 = -2c_N \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Lambda_{2(k-\sigma)}^{(N-1)} \Gamma_{2(\sigma+s)-1}^{(N)}, \tag{59}$$

$$\begin{aligned} \rho_4 = & -2c_N^2 \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k-1} \Gamma_{2(k-\sigma)-1}^{(N)} \Gamma_{2(\sigma+s)-1}^{(N)} \\ & -\alpha c_N \left[1-(-1)^N\right] \Gamma_{2s-1}^{(N)}, \end{aligned} \tag{60}$$

$$\begin{aligned} \rho_5 = & 2c_N c_{N-1} \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k-1} \Gamma_{2(k-\sigma)-1}^{(N-1)} \Gamma_{2(\sigma+s)-1}^{(N)} \\ & +\alpha c_N \left[1-(-1)^{N-1}\right] \Gamma_{2s-1}^{(N)}. \end{aligned} \tag{61}$$

For large–N matrix models with a single spectrum support there are asymptotic identities $c_N \approx c_{N-1}$, $\Lambda_\sigma^{(N)} \approx \Lambda_\sigma^{(N-1)}$ and $\Gamma_\sigma^{(N)} \approx \Gamma_\sigma^{(N-1)}$ (see Appendix A for details) which simplify matters greatly. Collecting Eqs. (57) – (61) we come down to

$$\int_{-\infty}^{+\infty} d\lambda \lambda^{2s} \nu_N(\lambda) \overset{N\to\infty}{\to} -2\alpha c_N (-1)^N \Gamma_{2s-1}^{(N)}$$

$$+ \qquad 2c_N \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Lambda^{(N)}_{2(k-\sigma)} \left(\Lambda^{(N)}_{2(\sigma+s-1)} - 2c_N \Gamma^{(N)}_{2(\sigma+s)-1} \right). \qquad (62)$$

Further double summation over indices k and σ is performed by making use of the integral representations for $\Lambda^{(N)}_{\sigma}$ and $\Gamma^{(N)}_{\sigma}$ given by Eqs. (129) and (131) in Appendix A. Straightforward calculations lead to

$$\int_{-\infty}^{+\infty} d\lambda \lambda^{2s} \nu_N(\lambda) \overset{N\to\infty}{\to} \int_{-\mathcal{D}_N}^{\mathcal{D}_N} d\lambda \lambda^{2s} \rho_{\Sigma}(\lambda), \qquad (63)$$

with $\mathcal{D}_N = 2c_N$ and

$$\begin{aligned} \rho_{\Sigma}(\lambda) &= \frac{2}{\pi^2} \left(\mathcal{D}_N^2 - \lambda^2 \right)^{1/2} \mathcal{P} \int_0^{\mathcal{D}_N} \frac{dt}{\left(\mathcal{D}_N^2 - t^2 \right)^{1/2}} \frac{t v'(t) - \lambda v'(\lambda)}{t^2 - \lambda^2} \\ &\quad - \frac{\alpha}{\pi} \frac{(-1)^N}{\left(\mathcal{D}_N^2 - \lambda^2 \right)^{1/2}}. \end{aligned} \qquad (64)$$

For $\lambda = \mathcal{D}_N z$ with $|z| < 1$ the term proportional to $\lambda dv/d\lambda$ vanishes identically due to the principal value $\mathcal{P}$ of the integral over variable t, while the term proportional to α is a subleading one in the large–N limit. Then, in accordance with our concept Eq. (44), we end up with

$$\nu_N^{\text{smooth}}(\lambda) = \frac{2}{\pi^2} \left(\mathcal{D}_N^2 - \lambda^2 \right)^{1/2} \mathcal{P} \int_0^{\mathcal{D}_N} \frac{t dt}{\left(\mathcal{D}_N^2 - t^2 \right)^{1/2}} \frac{dv/dt}{t^2 - \lambda^2}. \qquad (65)$$

Equation (65) is exactly the Dyson density $\nu_D(\lambda)$ of states with $\mathcal{D}_N$ being the end point of the eigenvalue support. We reconstructed the macroscopic level density Eq. (65) directly from the recurrence equation (28), alternatively to the traditional mean–field–theory derivation [1]. Notice that the spectrum end point $\mathcal{D}_N$ is the positive root to the integral equation

$$\frac{\pi N}{2} = \int_0^{\mathcal{D}_N} \frac{t dt}{\left(\mathcal{D}_N^2 - t^2 \right)^{1/2}} \frac{dv}{dt} \qquad (66)$$

following from the normalization of the level density.

4.2. GLOBAL CONNECTED "DENSITY–DENSITY" CORRELATION FUNCTION

The same technology is applicable to the study of the smoothed connected "density–density" correlator $\rho_c^{(N)}$. It is defined in terms of the scalar kernel

by Eq. (27), so that

$$\rho_c^{(N)}(\lambda,\lambda') = -\overline{K_N^2(\lambda,\lambda')} = -\frac{c_N^2}{(\lambda-\lambda')^2} \times\left[\overline{\left(\varphi_N^{(\alpha)}(\lambda)\right)^2}\,\overline{\left(\varphi_{N-1}^{(\alpha)}(\lambda')\right)^2} + \overline{\left(\varphi_{N-1}^{(\alpha)}(\lambda)\right)^2}\,\overline{\left(\varphi_N^{(\alpha)}(\lambda')\right)^2} -2\overline{\varphi_N^{(\alpha)}(\lambda)\,\varphi_{N-1}^{(\alpha)}(\lambda)}\;\overline{\varphi_N^{(\alpha)}(\lambda')\,\varphi_{N-1}^{(\alpha)}(\lambda')}\right]. \quad (67)$$

We remind that here $\varphi_N^{(\alpha)}(\lambda) = \exp\{-V_\alpha(\lambda)\}\,P_N^{(\alpha)}(\lambda)$ are fictitious wave functions, $\lambda \neq \lambda'$, and $\overline{(\ldots)}$ denotes averaging over rapid oscillations manifested on the characteristic scale of the mean level spacing. The averaging in Eq. (67) can be done along the lines of the previous Subsection with two modifications. First, as far as $\lambda \neq \lambda'$ we can run averaging over λ and λ' independently (this is already reflected in Eq. (67)). Second, we have to take into account the evenness of $\left(\varphi_N^{(\alpha)}(\lambda)\right)^2$ and the oddness of $\varphi_N^{(\alpha)}(\lambda)\,\varphi_{N-1}^{(\alpha)}(\lambda)$.

There are two integrals

$$I_1^{(N)} = \int_{-\infty}^{+\infty} d\lambda\lambda^{2s}\exp\{-2V_\alpha(\lambda)\}\left(P_{N-1}^{(\alpha)}(\lambda)\right)^2, \quad (68)$$

$$I_2^{(N)} = \int_{-\infty}^{+\infty} d\lambda\lambda^{2s+1}\exp\{-2V_\alpha(\lambda)\}\,P_N^{(\alpha)}(\lambda)\,P_{N-1}^{(\alpha)}(\lambda) \quad (69)$$

to be evaluated in the large–N limit. With the help of Eqs. (47) and (48) one immediately recognizes them as the objects $\Lambda_{2s}^{(N)}$ and $\Gamma_{2s+1}^{(N)}$, respectively, calculated in Appendix A. By comparing of Eqs. (68) and (69) with Eqs. (129) and (131) we deduce that

$$\overline{\left(\varphi_N^{(\alpha)}(\lambda)\right)^2} = \frac{1}{\pi}\left(\mathcal{D}_N^2-\lambda^2\right)^{-1/2}, \quad (70)$$

$$\overline{\varphi_N^{(\alpha)}(\lambda)\,\varphi_{N-1}^{(\alpha)}(\lambda)} = \frac{\lambda}{\pi\mathcal{D}_N}\left(\mathcal{D}_N^2-\lambda^2\right)^{-1/2}, \quad (71)$$

for $|\lambda| < \mathcal{D}_N$. Substituting Eqs. (70) and (71) into Eq. (67) is a final step leading us to the smoothed "density–density" correlator

$$\rho_c^{(N)}(\lambda,\lambda') = -\frac{1}{2\pi^2(\lambda-\lambda')^2}\frac{\mathcal{D}_N^2-\lambda\lambda'}{\left(\mathcal{D}_N^2-\lambda^2\right)^{1/2}\left(\mathcal{D}_N^2-\lambda'^2\right)^{1/2}} \quad (72)$$

announced by Eq. (2) with $\beta = 2$. We stress, that it has been obtained here from the recurrence equation for orthogonal polynomials associated with the random matrix ensemble in question.

4.3. LOCAL EIGENVALUE CORRELATIONS BY SOLVING EFFECTIVE SCHRÖDINGER EQUATION

4.3.1. *Effective Schrödinger Equation*

Local eigenvalue correlations in the matrix ensemble Eq. (4) can be studied by using the asymptotic version of the effective Schrödinger equation (36) obtained in Section 3. For the confinement potential V_α introduced by Eq. (41) we obtain in the large–N limit the following expressions for the functions $\mathcal{F}_n^{(\alpha)}(\lambda)$ and $\mathcal{G}_n^{(\alpha)}(\lambda)$ (see Eqs. (37) and (38)),

$$\mathcal{F}_N^{(\alpha)}(\lambda) = \frac{d}{d\lambda}\log A_N^{(\alpha)}(\lambda), \tag{73}$$

$$\begin{aligned}\mathcal{G}_N^{(\alpha)}(\lambda) &= \left(A_N^{(\alpha)}(\lambda)\right)^2\left[1-\left(\frac{\lambda}{\mathcal{D}_N}\right)^2\right] \\ &\quad + (-1)^N\frac{\alpha}{\lambda}\left(\frac{d}{d\lambda}\log A_N^{(\alpha)}(\lambda)\right) + \frac{\alpha(-1)^N-\alpha^2}{\lambda^2}.\end{aligned} \tag{74}$$

Here we have used the sum rule Eq. (39) to eliminate $B_N^{(\alpha)}$. For confinement potential with a smooth regular part v, the second term in Eq. (74) is a subleading and hence it must be discarded[5]. Note, that it is rather interesting that the confinement potential V_α does not appear in both equations above in an explicit way. It is even more exciting that in the considered approximation the function $A_N^{(\alpha)}$ can be solely expressed through the Dyson density, Eq. (65). Indeed, taking into account the representation Eq. (132) and the fact that $A_N^{(\alpha)}(\lambda) = A_{\rm reg}^{(N)}(\lambda)$, we are led to the asymptotic relation

$$A_N^{(\alpha)}(\lambda) = \frac{\pi\nu_D(\lambda)}{\left[1-(\lambda/\mathcal{D}_N)^2\right]^{1/2}}, \tag{75}$$

where, in accordance with Eq. (65),

$$\nu_D(\lambda) = \frac{2}{\pi^2}\mathcal{P}\int_0^{\mathcal{D}_N}\frac{tdt}{t^2-\lambda^2}\frac{dv}{dt}\left(\frac{\mathcal{D}_N^2-\lambda^2}{\mathcal{D}_N^2-t^2}\right)^{1/2}. \tag{76}$$

This allows us to arrive at the following remarkable effective one–particle Schrödinger equation for the wave functions

$$\varphi_N^{(\alpha)}(\lambda) = |\lambda|^\alpha P_N^{(\alpha)}(\lambda)\exp\{-v(\lambda)\} \tag{77}$$

[5]This is not the case for the *multicritical* correlations near the origin $\lambda = 0$. A detailed discussion of this important situation can be found in the very recent paper of Ref. [66].

of fictitious non–interacting fermions in the large–N limit [48, 53]

$$\frac{d^2\varphi_N^{(\alpha)}}{d\lambda^2} - \left[\frac{d}{d\lambda}\log\left(\frac{\pi\nu_D(\lambda)}{\left[1-(\lambda/\mathcal{D}_N)^2\right]^{1/2}}\right)\right]\frac{d\varphi_N^{(\alpha)}}{d\lambda} + \left(\pi^2\nu_D^2(\lambda) + \frac{(-1)^N\alpha-\alpha^2}{\lambda^2}\right)\varphi_N^{(\alpha)}(\lambda) = 0. \tag{78}$$

Also, due to Eq. (29), one can verify that the wave functions $\varphi_{N-1}^{(\alpha)}(\lambda)$ and $\varphi_N^{(\alpha)}(\lambda)$ of two successive quantum states are connected by the relationship

$$\frac{d\varphi_N^{(\alpha)}}{d\lambda} = \frac{\pi\nu_D(\lambda)}{\left[1-(\lambda/\mathcal{D}_N)^2\right]^{1/2}}\left(\varphi_{N-1}^{(\alpha)}(\lambda) - \frac{\lambda}{\mathcal{D}_N}\varphi_N^{(\alpha)}(\lambda)\right) + (-1)^N\frac{\alpha}{\lambda}\varphi_N^{(\alpha)}(\lambda). \tag{79}$$

Equations (78) and (79) serve as a general basis for the study of eigenvalue correlations in non–Gaussian random matrix ensembles in an *arbitrary spectral range*.

It is instructive to analyze them in the particular case of GUE, where the Dyson density of states is the celebrated semicircle,

$$\nu_D^{\mathrm{GUE}}(\lambda) = \pi^{-1}\left(\mathcal{D}_N^2-\lambda^2\right)^{1/2} \tag{80}$$

with $\mathcal{D}_N = (2N)^{1/2}$. The square–root law for $\nu_D^{\mathrm{GUE}}(\lambda)$ immediately removes the first derivative $d\varphi_N^{(\alpha)}/d\lambda$ in Eq. (78), providing us with the possibility to interpret the fictitious fermions as those confined by a quadratic potential ($\alpha = 0$). As far as the semicircle is a distinctive feature of density of states in GUE only, one will always obtain a first derivative in the effective Schrödinger equation for non–Gaussian unitary ensembles of random matrices. Therefore, fictitious non–interacting fermions associated with non–Gaussian ensembles of random matrices occur in a non–Hermitean quantum mechanics.

An interesting property of these equations is that they do not contain the regular part of confinement potential explicitly, but only involve the Dyson density ν_D (analytically continued on the entire real axis) and the spectrum end point $\mathcal{D}_N$. In contrast, the logarithmic singularity (that does not affect the Dyson density) introduces additional singular terms into Eqs. (78) and (79), changing significantly the behavior of the wave function $\varphi_N^{(\alpha)}$ near the origin $\lambda = 0$. The influence of the singularity decreases rather rapidly outward from the origin.

Structure of the effective Schrödinger equation leads us to the following statements [48] valid in the thermodynamic limit:

- *Eigenvalue correlations are stable with respect to non–singular deformations of the confinement potential.*

- In the random matrix ensembles with well behaved confinement potential *the knowledge of Dyson density* (that is rather crude one–point characteristics coinciding with the real density of states only in the spectrum bulk) *is sufficient to determine the genuine density of states, as well as the n–point correlation function, everywhere.*

The latter conclusion is rather unexpected since it considerably reduces the knowledge required for computing n–point correlators.

Effective Schrödinger equation obtained above enables us to examine in a unified way the local eigenvalue correlations in non–Gaussian ensembles with $\mathrm{U}(N)$ symmetry in different scaling limits. As we show below, it inevitably leads to the universal Bessel correlations in the origin scaling limit [47, 17, 53], to the universal sine correlations in the bulk scaling limit [36, 25, 38], and to the universal Airy correlations in the soft–edge scaling limit [53]. Corresponding scalar kernels are given by Eqs. (85), (87) and (94), respectively.

4.3.2. *Origin scaling limit and the universal Bessel law*
Origin scaling limit deals with the region of the spectrum close to $\lambda = 0$ where confinement potential displays the logarithmic singularity. In the vicinity of the origin the Dyson density can be taken as being approximately a constant, $\nu_D(0) = 1/\Delta_N(0)$, where $\Delta_N(0)$ is the mean level spacing at the origin in the absence of the logarithmic deformation of potential v. Within the framework of this approximation, Eq. (78) reads

$$\frac{d^2\varphi_N^{(\alpha)}}{d\lambda^2} + \left(\frac{\pi^2}{\Delta_N^2(0)} + \frac{(-1)^N\alpha - \alpha^2}{\lambda^2}\right)\varphi_N^{(\alpha)}(\lambda) = 0. \tag{81}$$

Solution to this equation that remains finite at $\lambda = 0$ can be expressed by means of Bessel functions

$$\varphi_{2N}^{(\alpha)}(\lambda) = a\sqrt{\lambda}J_{\alpha-1/2}\left(\frac{\pi\lambda}{\Delta(0)}\right), \tag{82}$$

$$\varphi_{2N+1}^{(\alpha)}(\lambda) = b\sqrt{\lambda}J_{\alpha+1/2}\left(\frac{\pi\lambda}{\Delta(0)}\right), \tag{83}$$

where a and b are constants to be determined later, and $\Delta(0) = \Delta_{2N}(0) \approx \Delta_{2N+1}(0)$. Inserting these solutions into Eq. (23) we find that the scalar kernel can be written down as

$$K_{2N}^{(\alpha)}(\lambda,\lambda') = c\frac{\sqrt{\lambda\lambda'}}{\lambda'-\lambda}\left[J_{\alpha+1/2}\left(\frac{\pi\lambda}{\Delta(0)}\right)J_{\alpha-1/2}\left(\frac{\pi\lambda'}{\Delta(0)}\right)\right.$$

$$-J_{\alpha+1/2}\left(\frac{\pi\lambda'}{\Delta(0)}\right)J_{\alpha-1/2}\left(\frac{\pi\lambda}{\Delta(0)}\right)\Big], \tag{84}$$

where the unknown factor c can be found from the requirement $K_{2N}^{(\alpha=0)}(\lambda,\lambda) = 1/\Delta(0)$. This immediately yields us the value $c = -\pi/\Delta(0)$. Defining now the scaled variable $s = \lambda_s/\Delta(0)$, we obtain that in the origin scaling limit the scalar kernel $K_{\rm orig}(s,s') = \lim_{N\to\infty}\lambda'_s K_{2N}^{(\alpha)}(\lambda_s,\lambda_{s'})$ takes the universal Bessel law,

$$K_{\rm orig}(s,s') = \frac{\pi}{2}(ss')^{1/2}\,\frac{J_{\alpha+1/2}(\pi s)\,J_{\alpha-1/2}(\pi s') - J_{\alpha-1/2}(\pi s)\,J_{\alpha+1/2}(\pi s')}{s-s'}. \tag{85}$$

Equation (85) is valid for arbitrary $\alpha > -1/2$, thus extending a recent proof [17] of universality of the Bessel kernel.

4.3.3. *Bulk scaling limit and the universal sine law*
Bulk scaling limit is associated with a spectrum range where the confinement potential is well behaved (that is far from the logarithmic singularity $\lambda = 0$), and where the density of states can be taken as being approximately a constant on the scale of a few levels. In accordance with this definition one has

$$K_{\rm bulk}(s,s') = \lim_{s,s'\to\infty} K_{\rm orig}(s,s'), \tag{86}$$

where s and s' should, nevertheless, remain far enough from the end point $\mathcal{D}_N$ of the spectrum support.

Taking this limit in Eq. (86), we arrive at the universal sine law

$$K_{\rm bulk}(s,s') = \frac{\sin[\pi(s-s')]}{\pi(s-s')} \tag{87}$$

deeply connected to the Wigner–Dyson level statistics [32].

4.3.4. *Soft–edge scaling limit and the universal Airy law*
Soft–edge scaling limit is relevant to the tail of eigenvalue support where crossover occurs from a nonzero density of states to a vanishing one. It is known [50, 67] that by tuning coefficients d_k which enter the regular part v of confinement potential (see Eq. (42)), one can obtain a macroscopic (Dyson) density of states which possesses a singularity of the type $\nu_D(\lambda) \propto \left(1-\lambda^2/\mathcal{D}_N^2\right)^{m+1/2}$ with the multicritical index $m = 0, 2, 4$, etc. (Odd indices m are inconsistent with our choice that the leading coefficient d_p, entering the regular component $v(\lambda)$ of confinement potential, be positive in order to keep a convergence of integral for partition function $\mathcal{Z}_N$ in Eq. (4)). It was shown in Ref. [53] within the Shohat method that as long

as the multicriticality of the order m is reached, the eigenvalue correlations in the vicinity of the soft edge become universal, and are independent of the particular potential chosen. The order m of the multicriticality is the only parameter which governs spectral correlations in the soft–edge scaling limit. Here, however, we restrict ourselves to a general confinement potential without tuning to the multicritical point, that corresponds to $m = 0$.

Let us move the spectrum origin to its end point $\mathcal{D}_N$, making the replacement

$$\lambda_s = \mathcal{D}_N \left[1 + (s/2) \left(2(\pi \mathcal{D}_N \mathcal{R}_N(1))^{-1} \right)^{2/3} \right], \tag{88}$$

that defines the *soft–edge scaling limit* provided $s \ll (\mathcal{D}_N \mathcal{R}_N(1))^{2/3} \propto N^{2/3}$. It is straightforward to show from Eqs. (78) and (79) that in the vicinity of the end point $\mathcal{D}_N$ the function $\widehat{\varphi}_N(s) = \varphi_N^{(\alpha)}(\lambda_s - \mathcal{D}_N)$ obeys the universal differential equation

$$\widehat{\varphi}_N''(s) - s\widehat{\varphi}_N(s) = 0, \tag{89}$$

and that the following relation takes place,

$$\widehat{\varphi}_{N-1}(s) = \widehat{\varphi}_N(s) + \left(\frac{2}{\pi \mathcal{D}_N \mathcal{R}_N(1)} \right)^{1/3} \widehat{\varphi}_N'(s). \tag{90}$$

Solution to Eq. (89) which decreases at $s \to +\infty$ (that is at far tails of the density of states) can be represented through the Airy function

$$\mathrm{Ai}(s) = \frac{1}{3} \begin{cases} s^{1/2} \left[I_{-1/3}\left(\frac{2}{3} s^{3/2} \right) - I_{1/3}\left(\frac{2}{3} s^{3/2} \right) \right], & s > 0, \\ |s|^{1/2} \left[J_{-1/3}\left(\frac{2}{3} |s|^{3/2} \right) + J_{1/3}\left(\frac{2}{3} |s|^{3/2} \right) \right], & s < 0. \end{cases} \tag{91}$$

as follows

$$\widehat{\varphi}_N(s) = a\mathrm{Ai}(s), \tag{92}$$

with a being an unknown constant. Making use of Eq. (90), we obtain that in the vicinity of the soft edge the scalar kernel is

$$K_N(\lambda_s, \lambda_{s'}) = b \frac{\mathrm{Ai}(s)\,\mathrm{Ai}'(s') - \mathrm{Ai}(s')\,\mathrm{Ai}'(s)}{s - s'}, \tag{93}$$

where b is an unknown constant again. It can be found by fitting [51] the density of states $K_N(\lambda_s, \lambda_s)$, Eq. (93), to the Dyson density of states $\nu_D(\lambda_s)$, Eq. (76), near the soft edge provided $1 \ll s \ll N^{2/3}$. This yields us the value $b = c_N^{-1}(\pi c_N \mathcal{R}_N(1))^{2/3}$. Thus, we obtain that in the soft–edge scaling limit, Eq. (88), the scalar kernel $K_{\mathrm{soft}}(s, s') = \lim_{N\to\infty} \lambda_s' K_N(\lambda_s, \lambda_{s'})$ satisfies the universal Airy law

$$K_{\mathrm{soft}}(s, s') = \frac{\mathrm{Ai}(s)\,\mathrm{Ai}'(s') - \mathrm{Ai}(s')\,\mathrm{Ai}'(s)}{s - s'} \tag{94}$$

which does not depend on the details of the confinement potential. In fact, the Airy law is a particular case ($m = 0$) of more general multicritical correlations characterized by the index m of the multicriticality. For more details we refer the reader to Refs. [53, 48].

It follows from Eq. (94) that the density of states in the same scaling limit

$$\nu_{\text{soft}}(s) = \left(\frac{d}{ds}\text{Ai}(s)\right)^2 - s\left[\text{Ai}(s)\right]^2 \tag{95}$$

is also universal. The large–$|s|$ behavior of ν_{soft} can be deduced from the known asymptotic expansions [68] of the Bessel functions,

$$\nu_{\text{soft}} = \begin{cases} \frac{|s|^{1/2}}{\pi} - \frac{1}{4\pi|s|}\cos\left(\frac{4}{3}|s|^{3/2}\right), & s \to -\infty, \\ \frac{1}{8\pi s}\exp\left(-\frac{4}{3}s^{3/2}\right), & s \to +\infty. \end{cases} \tag{96}$$

Note that the leading order behavior as $s \to -\infty$ is consistent with the $|s|^{1/2}$ singularity of the bulk density of states.

4.4. DISCUSSION

Looking back at the formalism developed we should reiterate that the crucial point in the derivations above is the large–N limit, Eq. (124), of the recurrence equation for associated orthogonal polynomials. It was precisely this limit that led us to the important relation Eq. (75) and to the effective Schrödinger equation in the form of Eq. (78) which is a nonuniversal in general. However, it takes locally universal forms in the spectrum bulk (where ν_D is approximatelly a constant on the scale of a few eigenlevels), near the spectrum origin (where all the nontrivial information is contained in λ^{-2} term in front of $\varphi_N^{(\alpha)}$ in Eq. (78)), and near the soft edge of the spectrum (where universality shows up in the universal square–root singularity of ν_D). These three locally universal features of Eq. (78) have led us to the universal sine, Bessel and Airy kernels in corresponding scaling limits.

We stress that Eq. (124) is, in fact, the leading–order–limit as $N \to \infty$. How accurate is this approximation, and do situations exist where the next–order terms in the recurrence equation should be taken into account? A partial answer to this question was given in Ref. [66] whose authors, remaining within the framework of the Shohat method, convincingly demonstrated that corrections to Eq. (124) are of importance in a problem of multicritical spectral correlations near the spectrum origin. The effective Schrödinger equation obtained there for two particular matrix ensembles with fine–tuned confinement potentials was shown not only to depend on the macroscopic spectral density $\nu_D(\lambda)$ but, in addition, to contain contributions from subdominant terms in $1/N$ expansion for the recurrence

coefficients. It is important however, that in the situation in question the resulting differential equation contained the universal functions $\mathcal{F}_N$ and $\mathcal{G}_N$ involving certain universal combinations of recurrence coefficients and coupling constants responsible for the fine tuning of the multicritical confinement potential. Having a complicated form, the effective Schrödinger equation could not be solved analytically, but, remarkably, the authors of Ref. [66] succeded in identifying a certain "mesoscopic" limit, in which the numerical solution of the exact differential equation and the analytical solution of the approximate differential equation obtained by making use of the relation Eq. (75) were shown to have quite similar qualitative features. With increasing of the order of the multicriticality near the spectrum origin, the approximate (analytical) and exact (numerical) solutions were shown to approach each other even quantitatively, demonstrating thus the potentialities of the Shohat method even in its simplest formulation presented above.

5. Two–Band Random Matrices

5.1. MULTI–BAND SPECTRAL REGIMES

Ensembles of large random matrices $\mathbf{H}$ generated by the joint distribution function $P[\mathbf{H}]$, Eq. (1), may display phase transitions under non–monotonic deformation of the confinement potential $V[\mathbf{H}]$. Different phases are characterized by topologically different arrangements of eigenvalues in random matrix spectra that may have multiple–band structure. Random matrices, whose spectra undergo phase transitions, appear in quantizing two–dimensional gravity [69, 70, 71], in the context of quantum chromodynamics [72, 73], as well as in some models of particles interacting in high dimensions [74]. Transition regimes realized in invariant random matrix ensembles have implications for a certain class of Calogero–Sutherland–Moser models [75]. These matrix models may also be applicable to chaotic systems having a forbidden gap in the energy spectrum.

It is convenient to parametrize the confinement potential $V(\lambda)$ entering Eq. (1) by a set of coupling constants $\{d\} = \{d_1, ..., d_p\}$,

$$V(\lambda) = \sum_{k=1}^{p} \frac{d_k}{2k} \lambda^{2k}, \; d_p > 0, \tag{97}$$

so that we may consider the phase transitions as occurring in $\{d\}$–space. Because the confinement potential is an even function, the associated random matrix model possesses so–called $Z2$–symmetry.

Variations of the coupling constants affect the Dyson density ν_D, that can be found by minimizing the free energy $F_N = -\log \mathcal{Z}_N$, Eq. (1), subject

to a normalization constraint $\int \nu_D(\lambda)\, d\lambda = N$,

$$\frac{dV}{d\lambda} - \mathcal{P}\int dt \frac{\nu_D(t)}{\lambda - t} = 0, \tag{98}$$

where $\mathcal{P}$ indicates a principal value of the integral. When all d_k are positive, so that confinement potential is monotonic, the spectral density ν_D has a single–band support, $\mathcal{N}_{\rm b} = 1$. Non–monotonic deformation of the confinement potential can be carried out by changing the signs of some of d_k ($k \neq p$). Such a *continuous* variation of coupling constants may lead, under certain conditions, to a *discontinuous* change of the topological structure of spectral density ν_D, when the eigenvalues $\{\lambda\}$ are arranged in $\mathcal{N}_{\rm b} > 1$ "allowed" bands separated by "forbidden" gaps.

The phase structure of Hermitean ($\beta = 2$) one–matrix model Eq. (1) has been studied in a number of works [76, 77, 78, 79], where the simplest examples of non–monotonic quartic and sextic confinement potentials have been examined. It has been found that there are domains in the phase space of coupling constants where only a particular solution for ν_D exists, and it has a fixed number $\mathcal{N}_{\rm b}$ of allowed bands. However, in some regions of the phase space, one can have more than one kind of solution of the saddle–point equation Eq. (98). In this situation, solutions with different number of bands $\mathcal{N}_{\rm b}^{(1)}, \mathcal{N}_{\rm b}^{(2)}, \ldots$ are present simultaneously. When such an overlap appears, one of the solutions, say $\mathcal{N}_{\rm b}^{(k)}$, has the lowest free energy $F_N^{(k)}$, and this solution is dominant, while the others are subdominant. Moreover, numerical calculations [78] showed that some special regimes exist in which the *bulk* spectral density obtained as a solution to the saddle–point equation Eq. (98) differs significantly from the genuine level density computed numerically within the framework of the orthogonal polynomial technique. It was then argued that such a genuine density of levels cannot be interpreted as a multi–band solution with an integer number of bands. A full understanding of this phenomenon is still absent.

Recently, interest was renewed in multi–band regimes in invariant random matrix ensembles. An analysis based on a loop equation technique [20, 55] showed that fingerprints of phase transitions appear not only in the Dyson density but also in the (universal) wide–range eigenvalue correlators, which in the multi–band phases differ from those known in the single–band phase [19, 36, 23]. A renormalization group approach developed in Ref. [80] supported the results found in Refs. [20, 55] for the particular case of two allowed bands, referring a new type of universal wide–range eigenlevel correlators to an additional attractive fixed point of a renormalization group transformation.

As it was already stressed in the Introduction, the method of loop equations [20, 55], used for a treatment of non–Gaussian, unitary invariant,

random matrix ensembles fallen in a multi–band phase, is only suitable for computing the global characteristics of spectrum. Therefore, an appropriate approach is needed capable of analyzing local characteristics of spectrum (manifested on the scale of a few eigenlevels). A possibility to probe the local properties of eigenspectrum is offered by the method of orthogonal polynomials. A step in this direction was taken in the paper [81], where an ansatz was proposed for large–N asymptotes of orthogonal polynomials associated with a random matrix ensemble having two allowed bands in its spectrum. Because the asymptotic formula proposed there is of the Plancherel–Rotach type [60], it is only applicable for studying eigenvalue correlations in the spectrum bulk and cannot be used for studying local correlations in an arbitrary spectrum range (for example, near the edges of two–band eigenvalue support).

Below we demonstrate that the Shohat method needs minimal modifications to allow a unified treatment of eigenlevel correlations in the unitary invariant $\mathrm{U}(N)$ matrix model ($\beta = 2$) with a forbidden gap. In particular, we will be able to study both the fine structure of local characteristics of the spectrum in different scaling limits and smoothed global spectral correlations. As is the single–band phase, the treatment presented below is based on the direct reconstruction of spectral correlations from the recurrence equation for the corresponding orthogonal polynomials.

5.2. EFFECTIVE SCHRÖDINGER EQUATION IN THE TWO–BAND PHASE AND LOCAL EIGENVALUE CORRELATIONS

Let us consider the situation when the confinement potential has two deep wells leading to the Dyson density supported on two disjoint intervals located symmetrically about the origin, $\mathcal{D}_N^- < |\lambda| < \mathcal{D}_N^+$. In this situation, the recurrence coefficients c_n entering Eq. (28) are known to be double–valued functions of the number n [69, 79]. This means that for $n = N \gg 1$ and in contrast with a single–band phase, one must distinguish between coefficients $c_{N\pm 2q} \approx c_N$ and coefficients $c_{N-1\pm 2q} \approx c_{N-1}$, belonging to two different, smooth (in index) sub–sequences; here, integer $q \sim \mathcal{O}\left(N^0\right)$. Bearing this in mind, the large–N version of recurrence equation Eq. (28) can be rewritten as

$$\left[\lambda^2 - \left(c_N^2 + c_{N-1}^2\right)\right] P_N(\lambda) = c_N c_{N-1} \left[P_{N-1}(\lambda) + P_{N+1}(\lambda)\right], \qquad (99)$$

whence the two analogues of the asymptotic expansion Eq. (125) can be obtained. They are given by Eqs. (135) and (136) of Appendix B. We notice that this is the only point crucial for extending the Shohat method to the double–well matrix models considered in this Section. These new expansions make it possible to compute the required functions $\mathcal{F}_N$ and $\mathcal{G}_N$

entering the differential equation Eq. (36) for fictitious wave functions in the limit $N \gg 1$.

In accordance with the general framework of the Shohat method, we have to compute two functions (compare with Eqs. (55) and (56))

$$A_N(\lambda) = 2c_N \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Lambda^{(N)}_{2(k-\sigma)} \lambda^{2\sigma-2}, \tag{100}$$

$$B_N(\lambda) = 2c_N \sum_{k=1}^{p} d_k \sum_{\sigma=1}^{k} \Gamma^{(N)}_{2(k-\sigma)-1} \lambda^{2\sigma-1}, \tag{101}$$

involving the objects $\Lambda^{(N)}_{2\sigma}$ and $\Gamma^{(N)}_{2\sigma+1}$, for which there exist the useful integral representations given by Eqs. (140) and (148) in Appendix B. Substituting them into Eqs. (100) and (101) one is able to perform the double summation over indices k and σ. Omiting details of straightforward calculations we present the final answer given by the formulas

$$A_N(\lambda) = \frac{2}{\pi}\left(\mathcal{D}_N^+ - (-1)^N \mathcal{D}_N^-\right) \mathcal{P} \int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dV}{dt} \frac{t^2}{t^2-\lambda^2} \times \frac{dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2} \left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}}, \tag{102}$$

$$B_N(\lambda) = \frac{2\lambda}{\pi} \mathcal{P} \int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dV}{dt} \frac{t^2 - (-1)^N \mathcal{D}_N^- \mathcal{D}_N^+}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2} \left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}} \frac{dt}{t^2-\lambda^2} - \frac{dV}{d\lambda}. \tag{103}$$

Note, that along with a different (compared to the single–band phase) functional form of the functions $A_N(\lambda)$ and $B_N(\lambda)$, these functions are, in fact, double–valued in index N, and behave in a different way for odd and even N. This is a direct consequence of the "period–two" behavior [69, 79] of the recurrence coefficients c_n.

Having obtained the explicit expressions for functions A_N and B_N, it is easy to verify that coefficients $\mathcal{F}_n(\lambda)$ and $\mathcal{G}_n(\lambda)$ entering the differential equation Eq. (36) for the fictitious wave function $\varphi_n(\lambda)$ may be expressed in terms of the Dyson density $\nu_D^{(\mathrm{II})}$ in the two–cut phase supported on two disconnected intervals $\lambda \in \left(-\mathcal{D}_N^+, -\mathcal{D}_N^-\right) \cup \left(\mathcal{D}_N^-, \mathcal{D}_N^+\right)$,

$$\nu_D^{(\mathrm{II})}(\lambda) = \frac{2}{\pi^2} |\lambda| \, \mathcal{P} \int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} dt \frac{dV/dt}{t^2-\lambda^2} \left(\frac{\left(\mathcal{D}_N^+\right)^2 - \lambda^2}{\left(\mathcal{D}_N^+\right)^2 - t^2}\right)^{1/2} \left(\frac{\lambda^2 - \left(\mathcal{D}_N^-\right)^2}{t^2 - \left(\mathcal{D}_N^-\right)^2}\right)^{1/2} \tag{104}$$

when $N \gg 1$. This formula can be obtained either via the procedure of the Sec. 4.1 or within the mean–field approach, Eq. (98). Here $\mathcal{D}_N^-$ and $\mathcal{D}_N^+$ are the end points of the eigenvalue support that obey the two integral equations

$$\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dV}{dt} \frac{t^2 dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2} \left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}} = \frac{\pi N}{2}, \tag{105}$$

$$\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dV}{dt} \frac{dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2} \left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}} = 0, \tag{106}$$

derived in Appendix C.

By making use of the Eqs. (37), (38) and (102) – (104) we obtain in the large–N limit

$$\mathcal{F}_N(\lambda) = \frac{d}{d\lambda} \log \left(\frac{\pi |\lambda| \nu_D^{(\mathrm{II})}(\lambda)}{\left[\left(\mathcal{D}_N^+\right)^2 - \lambda^2\right]^{1/2} \left[\lambda^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}} \right), \tag{107}$$

$$\mathcal{G}_N(\lambda) = \left(\pi \nu_D^{(\mathrm{II})}(\lambda)\right)^2, \tag{108}$$

so that for $N \gg 1$ the effective Schrödinger equation in the two–cut phase reads [56]

$$\begin{aligned} \frac{d^2\varphi_N(\lambda)}{d\lambda^2} &- \left[\frac{d}{d\lambda} \log \left(\frac{\pi |\lambda| \nu_D^{(\mathrm{II})}(\lambda)}{\left[\left(\mathcal{D}_N^+\right)^2 - \lambda^2\right]^{1/2} \left[\lambda^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}} \right) \right] \frac{d\varphi_N(\lambda)}{d\lambda} \\ &+ \left(\pi \nu_D^{(\mathrm{II})}(\lambda)\right)^2 \varphi_N(\lambda) = 0. \end{aligned} \tag{109}$$

As $\mathcal{D}_N^-$ tends to zero, we reproduce the equation (78) with $\alpha = 0$ valid in the single–band regime.

Local eigenvalue correlations in the spectra of two–band random matrices are completely determined by the Dyson density of states entering the effective Schrödinger equation Eq. (109).

(i) In the spectrum bulk, the Dyson density is a well behaved function that can be taken approximately as being a constant on the scale of a few

eigenlevels. Then, in the vicinity of some λ_0 that is chosen to be far enough from the spectrum end points $\pm\mathcal{D}_N^{\pm}$, Eq. (109) takes the form

$$\frac{d^2\varphi_N(\lambda)}{d\lambda^2} + [\pi/\Delta(\lambda_0)]^2 \varphi_N(\lambda) = 0, \tag{110}$$

with $\Delta(\lambda_0) = 1/\nu_D^{(\mathrm{II})}(\lambda_0)$ being the mean level spacing in the vicinity of λ_0. Clearly, the universal sine law, Eq. (87), for the two–point kernel follows immediately.

(ii) To study the eigenvalue correlations near the end points of an eigenvalue support we notice that in the absence of the fine tunning of confinement potential, the Dyson density has a universal square–root singularity in the vicinity of $|\lambda| = \mathcal{D}_N^{\pm}$, that is $\nu_D^{(\mathrm{II})}(\lambda) \propto \left(1 - \left(\lambda/\mathcal{D}_N^{\pm}\right)^2\right)^{1/2}$. We then readily recover the universal Airy correlations, Eq. (94), previously found in the soft–edge scaling limit for $\mathrm{U}(N)$ invariant matrix model in the single–band phase.

5.3. GLOBAL CONNECTED "DENSITY–DENSITY" CORRELATOR

Let us turn to the study of the smoothed connected "density–density" correlator that is expressed in terms of the scalar kernel as follows (see Eq. (27)),

$$\begin{aligned}\rho_{c\mathrm{II}}^{(N)}(\lambda,\lambda') &= -\frac{c_N^2}{(\lambda-\lambda')^2}\left\{\overline{\varphi_N^2(\lambda)}\,\overline{\varphi_{N-1}^2(\lambda')} + \overline{\varphi_N^2(\lambda')}\,\overline{\varphi_{N-1}^2(\lambda)}\right. \\ &\left. -2\overline{\varphi_N(\lambda)\varphi_{N-1}(\lambda)}\,\overline{\varphi_N(\lambda')\varphi_{N-1}(\lambda')}\right\}. \end{aligned}\tag{111}$$

Here $\lambda \neq \lambda'$. Equation (111) contains (before averaging) rapid oscillations on the scale of the mean level spacing. These oscillations are due to presence of oscillating wave functions φ_N and φ_{N-1}.

To average over the rapid oscillations, we integrate, over the entire real axis, rapidly varying wave functions in Eq. (111) multiplied by an arbitrary, smooth, slowly varying function, which without any loss of generality can be choosen to be λ^{2s} for $\varphi_N^2(\lambda)$ and λ^{2s+1} for $\varphi_N(\lambda)\varphi_{N-1}(\lambda)$ (s is an arbitrary positive integer, $s > 0$). Consider, first, the integral

$$I_1^{(N)} = \int_{-\infty}^{+\infty} d\lambda \lambda^{2s} \varphi_N^2(\lambda) = \Lambda_{2s}^{(N)}. \tag{112}$$

With the help of Eq. (140), and bearing in mind that $\varphi_N^2(\lambda)$ is an even function, we conclude that

$$I_1^{(N)} = \frac{1}{\pi}\int_{\mathcal{D}_N^-<|\lambda|<\mathcal{D}_N^+} \frac{|\lambda|\,\lambda^{2s}d\lambda}{\left[\left(\mathcal{D}_N^+\right)^2-\lambda^2\right]^{1/2}\left[\lambda^2-\left(\mathcal{D}_N^-\right)^2\right]^{1/2}}, \tag{113}$$

whence, in the large–N limit,

$$\overline{\varphi_N^2(\lambda)} = \frac{\Omega_\lambda}{\pi}\frac{|\lambda|}{\left[\left(\mathcal{D}_N^+\right)^2-\lambda^2\right]^{1/2}\left[\lambda^2-\left(\mathcal{D}_N^-\right)^2\right]^{1/2}}. \tag{114}$$

Here

$$\Omega_\lambda = \Theta\left(\mathcal{D}_N^+-|\lambda|\right)\Theta\left(|\lambda|-\mathcal{D}_N^-\right) \tag{115}$$

with Θ being a step function. The same procedure should be carried out with the expression $\varphi_N(\lambda)\varphi_{N-1}(\lambda)$ in Eq. (111). Since this construction is an odd function of λ, we have to consider the integral

$$I_2^{(N)} = \int_{-\infty}^{+\infty} d\lambda\lambda^{2s+1}\varphi_N(\lambda)\varphi_{N-1}(\lambda) = \Gamma_{2s+1}^{(N)}. \tag{116}$$

With the help of Eq. (148), and exploiting the oddness of $\varphi_N(\lambda)\varphi_{N-1}(\lambda)$, we rewrite Eq. (116) in the form

$$\begin{aligned} I_2^{(N)} &= \frac{1}{\pi\left[\mathcal{D}_N^+-(-1)^N\mathcal{D}_N^-\right]} \\ &\times\int_{\mathcal{D}_N^-<|\lambda|<\mathcal{D}_N^+}\frac{\left[\lambda^2-(-1)^N\mathcal{D}_N^-\mathcal{D}_N^+\right]\mathrm{sgn}(\lambda)\,d\lambda}{\left[\left(\mathcal{D}_N^+\right)^2-\lambda^2\right]^{1/2}\left[\lambda^2-\left(\mathcal{D}_N^-\right)^2\right]^{1/2}}, \end{aligned} \tag{117}$$

whence

$$\begin{aligned} \overline{\varphi_N(\lambda)\varphi_{N-1}(\lambda)} &= \frac{\Omega_\lambda\,\mathrm{sgn}(\lambda)}{\pi\left[\mathcal{D}_N^+-(-1)^N\mathcal{D}_N^-\right]} \\ &\times\frac{\lambda^2-(-1)^N\mathcal{D}_N^-\mathcal{D}_N^+}{\left[\left(\mathcal{D}_N^+\right)^2-\lambda^2\right]^{1/2}\left[\lambda^2-\left(\mathcal{D}_N^-\right)^2\right]^{1/2}}. \end{aligned} \tag{118}$$

Combining Eqs. (111), (114), (118) and (147), we finally arrive at the following formula for smoothed "density–density" correlator [56]

$$\begin{aligned} \rho_{\mathrm{cII}}^{(N)}(\lambda,\lambda') &= -\frac{\mathrm{sgn}(\lambda\lambda')}{2\pi^2}\frac{\Omega_\lambda\Omega_{\lambda'}}{Q_N(\lambda)Q_N(\lambda')}\left\{(-1)^N\mathcal{D}_N^-\mathcal{D}_N^+\right. \\ &\left.+\frac{1}{(\lambda-\lambda')^2}\left[\lambda\lambda'-(\mathcal{D}_N^-)^2\right]\left[(\mathcal{D}_N^+)^2-\lambda\lambda'\right]\right\}, \end{aligned} \tag{119}$$

where

$$Q_N(\lambda) = \left[(\mathcal{D}_N^+)^2 - \lambda^2\right]^{1/2} \left[\lambda^2 - (\mathcal{D}_N^-)^2\right]^{1/2}. \tag{120}$$

It is seen from Eq. (120) that smoothed "density–density" correlator in the two–band phase differs from that in the single–band phase, Eq. (72). However, it is still universal in the sense that the information of the distribution Eq. (1) is encoded into the "density–density" correlator only through the end points $\mathcal{D}_N^{\pm}$ of the eigenvalue support. The striking parity effect in the new universal function Eq. (120), that is the *sharp* dependence of correlations on the oddness/evenness of the dimension N of the random matrices, is the main *qualitative* difference as compared to the global corelations in random matrices fallen in the single–band phase. This effect is most pronounced in the case of unbounded spectrum. The origin of this unusual large–N behavior will be discussed later on.

Finally, let us speculate about the universal correlator Eq. (120) in the limit of *unbounded* spectrum, $\mathcal{D}_N^+ \to \infty$, with a gap. Inasmuch as it describes correlations between the eigenlevels which are repelled from each other in accordance with the logarithmic law, that is known to be realized [82, 83] in weakly disordered systems on the energy scale $|\lambda - \lambda'| \ll E_c$ (E_c is the Thouless energy), we may conjecture that the corresponding limiting universal expression

$$\begin{aligned} \lim_{\mathcal{D}_N^+ \to +\infty} \rho_{\mathrm{cII}}^{(N)}(\lambda, \lambda') &= -\frac{\mathrm{sgn}(\lambda\lambda')}{2\pi^2 (\lambda - \lambda')^2} \Theta(|\lambda| - \Delta)\, \Theta(|\lambda'| - \Delta) \\ &\quad \times \frac{\lambda\lambda' - \Delta^2}{[\lambda^2 - \Delta^2]^{1/2} [\lambda'^2 - \Delta^2]^{1/2}}, \end{aligned} \tag{121}$$

reflects the universal properties of real chaotic systems with a forbidden gap $\Delta = \mathcal{D}_N^-$ and broken time reversal symmetry, provided $|\lambda - \lambda'| \ll E_c$. In two limiting situations (i) of gapless spectrum, $\Delta = 0$, and (ii) far from the gap, $|\lambda|, |\lambda'| \gg \Delta$, the correlator Eq. (121) coincides with that known in the Random Matrix Theory of gapless ensembles [36, 23] and derived in Ref. [82] within the framework of diagrammatic technique for spectrum of electron in a random impurity potential.

5.4. DISCUSSION

In this Section we have demonstrated how the Shohat method should be transformed in order to study both global and local spectral characteristics of $\mathrm{U}(N)$ invariant ensembles of large random matrices possessing $Z2$–symmetry, and deformed in such a way that their spectra contain a forbidden gap. We proved that in the pure two–band phase, the local eigenvalue

correlations are insensitive to this deformation both in the bulk and soft–edge scaling limits. In contrast, global smoothed eigenvalue correlations in the two–band phase differ drastically from those in the single–band phase, and generically satisfy a universal law, Eq. (120), which is unusually sensitive to the oddness/evenness of the random matrix dimension provided the spectrum support is bounded. On the formal level, this sensitivity is a direct consequence of the "period–two" behavior [69, 79] of the recurrence coefficients c_n that is characteristic of two–band phase of reduced Hermitean matrix model. To see this, consider the simplest connected correlator $\langle \mathrm{Tr}\, \mathbf{H}\, \mathrm{Tr}\, \mathbf{H} \rangle_c$ that can be *exactly* represented in terms of recurrence coefficients for any n,

$$\langle \mathrm{Tr}\, \mathbf{H}\, \mathrm{Tr}\, \mathbf{H} \rangle_c = c_n^2. \tag{122}$$

Since in the two–band phase c_n is a double–valued function of index n, alternating between two different functions as n goes from odd to even, the large–N limit of the correlator $\langle \mathrm{Tr}\, \mathbf{H}\, \mathrm{Tr}\, \mathbf{H} \rangle_c$ strongly depends on whether infinity is approached through odd or even N. Then, an implementation of a double–valued behavior of c_n into the higher order correlators of the form $\left\langle \mathrm{Tr}\, \mathbf{H}^k\, \mathrm{Tr}\, \mathbf{H}^l \right\rangle_c$ contributing to the connected "density–density" correlator gives rise to the universal expression Eq. (120), which is valid for the two–band random matrix model with pure Z2–symmetry.

Let us, however, point out that no such sensitivity has been detected in a number of previous studies [20, 55] exploiting a loop equation technique. One possible explanation comes from the following reasons. In the method of loop equations, used for a treatment of non–Gaussian random matrix ensembles fallen in a multi–band phase, one has to keep the most general (non–symmetric) confinement potential $V(\lambda) = \sum_{k=1}^{2p} \widetilde{d}_k \lambda^k / k$ until very end of the calculations, and to take the thermodynamic limit $N \to \infty$ prior to any others. Therefore, $Z2$–symmetry in this calculational scheme can only be implemented by restoring $Z2$–symmetry at the final stage of the calculations, setting all the extra coupling constants $\widetilde{d}_{2k+1}$ to zero. Doing so, one arrives at the results reported in Refs. [20, 55]. From this point of view, our treatment corresponds to the *opposite sequence* of thermodynamic and $Z2$–symmetry limits, since we have considered the random matrix model that possesses $Z2$–symmetry from the beginning. Qualitatively different large–N behavior of the smoothed connected "density–density" correlator, Eq. (120), and of the smoothed connected two–point Green's function given by Eq. (15) of Ref. [55] provides a direct evidence that the order of thermodynamic and $Z2$–symmetry limits is indeed important when studying global spectral characteristics of multi–band random matrices.

The parity effect manifested in global spectral correlators of double–well matrix models was the focus of the discussion in the recent study [84].

The authors of Ref. [84] noted that, contrary to the method of orthogonal polynomials, the standard large–N limit techniques of analyzing matrix models like the loop equation method [19, 20] and the renormalization group approach [85] assume a smooth behavior with respect to N in the thermodynamic limit. The result Eq. (120) obtained by the authors of Ref. [84] in a different way [81] led them to conclusion that these methods need to be revisited when one deals with matrix models posessing eigenvalue gaps.

6. Conclusion

In this review we presented a formalism for statistical description of spectra of $\mathrm{U}(N)$ invariant ensembles of large random matrices. It lies within the general framework of the orthogonal polynomials' technique, and consists of the direct reconstruction of spectral densities and spectral correlations from the recurrence equation for orthogonal polynomials associated with a given random matrix ensemble. We have demonstrated the potentialities of this method, considering in a unified way both global and local spectral characteristics in matrix models with and without an eigenvalue gap. Although we directed our main attention to the most known bulk, origin and soft–edge scaling limits characterized by the universal sine, Bessel and Airy kernels, respectively, there are examples in the recent literature signaling about applicability of the described formalism to more refined situations – such as multicritical correlations near the soft edge of the spectrum support [53] and near the spectrum origin [66].

Attaching special significance to the study of the large–N limit of the recurrence equation for associated orthogonal polynomials, this method turns the recurrence equation into a kind of laboratory allowing the construction of matrix models with nonstandard properties – for example with eigenvalue gaps – by guessing a particular ansatz for the behavior of the recurrence coefficients c_N in the thermodynamic limit. Just this feature of the formalism presented forces us, finally, to mention a crucial difference between the random matrix ensembles with strong level confinement considered here and the random matrix ensembles with extremely soft level confinement. While the former ensembles (with confinement potentials of the Freud and Erdös type [38]) are characterized by a powerlike large–N limit of recurrence coefficients, $c_N \propto N^{\rho}$ $(\rho > 0)$, the latter (representing a class of q–deformed potentials [86, 87]) exhibit a qualitatively different, exponential rate of growth, $c_N \propto q^N$ $(q > 1)$. This results in a different large–N limit of the recurrence equation that will not already be as simple as stated in Eq. (124), and the emergence of different nontrivial classes of spectral statistics is inevitable. We consider a treatment of q–deformed ran-

dom matrix ensembles as a challenge to the Shohat method which, going back to 1930, had to wait so long to find its application in Random Matrix Theory.

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A. Integral Representation of $\Lambda_{2\sigma}^{(N)}$ and $\Gamma_{2\sigma+1}^{(N)}$: Single–Band Phase

Consider the integral

$$\Lambda_{2\sigma}^{(N)} = \int d\alpha\,(t)\left(P_N^{(\alpha)}\,(t)\right)^2 t^{2\sigma} \tag{123}$$

with integer $\sigma \geq 0$. In the large–N limit an alternative explicit integral representation can be found for $\Lambda_{2\sigma}^{(N)}$. This is achieved by making use of the large–N version of the recurrence equation (28). It is known that in the single–band phase of the matrix model the recurrence coefficients approach [69, 79] a smooth (in index N) single–valued function, so that $c_{N+q} \approx c_N$ for q being of order $\mathcal{O}\left(N^0\right)$. Within this approximation one obtains from Eq. (28)

$$\lambda P_N^{(\alpha)}\,(\lambda) = c_N\left(P_{N-1}^{(\alpha)}\,(\lambda) + P_{N+1}^{(\alpha)}\,(\lambda)\right), \tag{124}$$

whence it follows that

$$\lambda^m P_N^{(\alpha)}\,(\lambda) = c_N^m \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) P_{N+2j-m}^{(\alpha)}\,(\lambda)\,,\ m \geq 1. \tag{125}$$

The identity (125) can be proven by the mathematical induction. The advantage of the asymptotic expansion Eq. (125) is that being substituted (for $m = 2\sigma$) into Eq. (123), it immediately allows us to explicitly perform the integration due to the orthogonality property Eq. (17). This yields

$$\Lambda_{2\sigma}^{(N)} = c_N^{2\sigma} \sum_{j=0}^{2\sigma} \left(\begin{array}{c} 2\sigma \\ j \end{array} \right) \delta_{2j}^{2\sigma}\,, \tag{126}$$

with δ_j^k being the Kronecker symbol. To evaluate the sum in Eq. (126) we make use of the integral representation of the Kronecker symbol,

$$\delta_j^k = \mathrm{Re} \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left\{i\theta\,(j-k)\right\}. \tag{127}$$

We then find that

$$\Lambda_{2\sigma}^{(N)} = c_N^{2\sigma}\,\mathrm{Re}\int_0^{2\pi}\frac{d\theta}{2\pi}\sum_{j=0}^{2\sigma}\left(\begin{array}{c}2\sigma\\ j\end{array}\right)\exp\left\{2i\theta\left(j-\sigma\right)\right\}, \tag{128}$$

and hence, after summation over j and some rearrangements,

$$\Lambda_{2\sigma}^{(N)} = \frac{2}{\pi}\int_0^{\mathcal{D}_N}\frac{dt\,t^{2\sigma}}{\left(\mathcal{D}_N^2-t^2\right)^{1/2}}. \tag{129}$$

Here $\mathcal{D}_N = 2c_N$.

The integral

$$\Gamma_{2\sigma+1}^{(N)} = \int d\alpha\left(t\right)P_N^{(\alpha)}\left(t\right)P_{N-1}^{(\alpha)}\left(t\right)t^{2\sigma+1} \tag{130}$$

with integer $\sigma \geq 0$ is computable in the same manner, with an answer given by

$$\Gamma_{2\sigma+1}^{(N)} = \frac{2}{\pi\mathcal{D}_N}\int_0^{\mathcal{D}_N}\frac{dt\,t^{2\sigma+2}}{\left(\mathcal{D}_N^2-t^2\right)^{1/2}}. \tag{131}$$

Notice that due to the asymptotic property $c_{N+q} \approx c_N$ mentioned above, $\Lambda_{2\sigma}^{(N+q)} \approx \Lambda_{2\sigma}^{(N)}$ and $\Gamma_{2\sigma+1}^{(N+q)} \approx \Gamma_{2\sigma+1}^{(N)}$ for $q \sim \mathcal{O}\left(N^0\right)$.

Finally, we demonstrate the usefulness of the formulas (129) and (131) by finding the explicit expressions for the functions $A_{\mathrm{reg}}^{(N)}\left(\lambda\right)$ and $B_{\mathrm{reg}}^{(N)}\left(\lambda\right)$, defined by Eqs. (49) and (51). Substitution of Eq. (129) into Eq. (49) followed by summation over σ yields

$$\begin{aligned} A_{\mathrm{reg}}^{(N)}\left(\lambda\right) &= \frac{2\mathcal{D}_N}{\pi}\mathcal{P}\int_0^{\mathcal{D}_N}\frac{dt}{\left(\mathcal{D}_N^2-t^2\right)^{1/2}}\sum_{k=1}^{p}d_k\frac{t^{2k}-\lambda^{2k}}{t^2-\lambda^2} \\ &= \frac{2\mathcal{D}_N}{\pi}\mathcal{P}\int_0^{\mathcal{D}_N}\frac{dt}{\left(\mathcal{D}_N^2-t^2\right)^{1/2}}\frac{tdv/dt}{t^2-\lambda^2}. \end{aligned} \tag{132}$$

Analogously, we obtain

$$\begin{aligned} B_{\mathrm{reg}}^{(N)}\left(\lambda\right) &= \frac{2}{\pi}\mathcal{P}\int_0^{\mathcal{D}_N}\frac{tdt}{\left(\mathcal{D}_N^2-t^2\right)^{1/2}}\frac{\lambda dv/dt - tdv/d\lambda}{t^2-\lambda^2} \\ &= \frac{2\lambda}{\pi}\mathcal{P}\int_0^{\mathcal{D}_N}\frac{dt}{\left(\mathcal{D}_N^2-t^2\right)^{1/2}}\frac{tdv/dt}{t^2-\lambda^2} - \frac{dv}{d\lambda}. \end{aligned} \tag{133}$$

One can convince himself that Eqs. (132) and (133) obey the sum rule Eq. (39) for the confinement potential V_α, Eq. (41), and for the functions $A_N^{(\alpha)}$

and $B_N^{(\alpha)}$, given by Eqs. (55) and (56). Notice that this is the recurrence equation (28) which enabled us to obtain closed analytic expressions (132) and (133) relating the functions $A_{\rm reg}^{(N)}$ and $B_{\rm reg}^{(N)}$ to the regular part v of the confinement potential.

B. Integral Representation of $\Lambda_{2\sigma}^{(N)}$ and $\Gamma_{2\sigma+1}^{(N)}$: Two–Band Phase

Consider the integral

$$\Lambda_{2\sigma}^{(N)} = \int d\alpha\,(t)\,P_N^2\,(t)\,t^{2\sigma} \tag{134}$$

with integer $\sigma \geq 0$. It follows from Eq. (99) that in the two–band phase the following asymptotic identities exist,

$$\begin{aligned}\lambda^{2m} P_N\,(\lambda) &= \left(c_N^2 + c_{N-1}^2\right)^m \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) \left(\frac{c_N c_{N-1}}{c_N^2 + c_{N-1}^2} \right)^k \\ &\times \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) P_{N+4j-2k}\,(\lambda)\,, \end{aligned} \tag{135}$$

and

$$\begin{aligned}\lambda^{2m+1} P_N\,(\lambda) &= \left(c_N^2 + c_{N-1}^2\right)^m \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) \left(\frac{c_N c_{N-1}}{c_N^2 + c_{N-1}^2} \right)^k \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \\ &\times \left[c_{N-1} P_{N+4j-2k+1}\,(\lambda) + c_N P_{N+4j-2k-1}\,(\lambda) \right] \end{aligned} \tag{136}$$

with integer $m \geq 0$. Both Eqs. (135) and (136) can be proven by the mathematical induction. Making use of Eq. (135) we rewrite $\Lambda_{2\sigma}^{(N)}$ in the form

$$\begin{aligned}\Lambda_{2\sigma}^{(N)} &= \left(c_N^2 + c_{N-1}^2\right)^\sigma \sum_{k=0}^{\sigma} \left(\begin{array}{c} \sigma \\ k \end{array} \right) \left(\frac{c_N c_{N-1}}{c_N^2 + c_{N-1}^2} \right)^k \\ &\times \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \int d\alpha\,(t)\,P_N\,(t)\,P_{N+4j-2k}\,(t)\,. \end{aligned} \tag{137}$$

Orthogonality of P_n allows us to integrate over the measure $d\alpha$, thus simplifying Eq. (137) to

$$\Lambda_{2\sigma}^{(N)} = \left(c_N^2 + c_{N-1}^2\right)^\sigma \sum_{k=0}^{\sigma} \left(\begin{array}{c} \sigma \\ k \end{array} \right) \left(\frac{c_N c_{N-1}}{c_N^2 + c_{N-1}^2} \right)^k \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \delta_{2j}^k. \tag{138}$$

Substituting the integral representation (127) for the Kronecker symbol, and performing the double summation over indices j and k, we obtain

$$\Lambda_{2\sigma}^{(N)} = \int_0^{2\pi} \frac{d\theta}{2\pi} \left(c_N^2 + c_{N-1}^2 + 2c_N c_{N-1} \cos\theta \right)^{\sigma} . \tag{139}$$

Introducing a new integration variable $t^2 = c_N^2 + c_{N-1}^2 + 2c_N c_{N-1} \cos\theta$, we derive the final formula

$$\Lambda_{2\sigma}^{(N)} = \frac{2}{\pi} \int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{t^{2\sigma+1} dt}{\left[\left(\mathcal{D}_N^+ \right)^2 - t^2 \right]^{1/2} \left[t^2 - \left(\mathcal{D}_N^- \right)^2 \right]^{1/2}} \tag{140}$$

with

$$\mathcal{D}_N^{\pm} = |c_N \pm c_{N-1}| . \tag{141}$$

The integral

$$\Gamma_{2\sigma+1}^{(N)} = \int d\alpha\,(t)\, P_N\,(t)\, P_{N-1}\,(t)\, t^{2\sigma+1} \tag{142}$$

with integer $\sigma \geq 0$ is evaluated in the same way. Making use of expansion Eq. (136), we rewrite Eq. (142) in the form that allows us to perform the integration over the measure $d\alpha$,

$$\begin{aligned} \Gamma_{2\sigma+1}^{(N)} &= \frac{1}{2} \left(c_N^2 + c_{N-1}^2 \right)^{\sigma} \int d\alpha\,(t)\, P_{N-1}\,(t) \sum_{k=0}^{\sigma} \left(\begin{array}{c} \sigma \\ k \end{array} \right) \left(\frac{c_N c_{N-1}}{c_N^2 + c_{N-1}^2} \right)^k \\ &\times \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \left[c_{N-1} P_{N+4j-2k+1}\,(t) + c_N P_{N+4j-2k-1}\,(t) \right] . \end{aligned} \tag{143}$$

After integration, we get

$$\begin{aligned} \Gamma_{2\sigma+1}^{(N)} &= \frac{1}{2} \left(c_N^2 + c_{N-1}^2 \right)^{\sigma} \sum_{k=0}^{\sigma} \left(\begin{array}{c} \sigma \\ k \end{array} \right) \left(\frac{c_N c_{N-1}}{c_N^2 + c_{N-1}^2} \right)^k \\ &\times \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \left[c_{N-1} \delta_{2j+1}^k + c_N \delta_{2j}^k \right] . \end{aligned} \tag{144}$$

The double summation in Eq. (144) can be performed by using the integral representation for the Kronecker symbol given by Eq. (127),

$$\Gamma_{2\sigma+1}^{(N)} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{2\pi} \left(c_N^2 + c_{N-1}^2 + 2c_N c_{N-1} \cos\theta \right)^{\sigma} \left[c_N + c_{N-1} \cos\theta \right] . \tag{145}$$

Introducing a new integration variable $t^2 = c_N^2 + c_{N-1}^2 + 2c_N c_{N-1}\cos\theta$, we get

$$\Gamma_{2\sigma+1}^{(N)} = \frac{1}{\pi c_N}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{t^{2\sigma+1}dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2}\left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}}\left[t^2 + c_N^2 - c_{N-1}^2\right]. \tag{146}$$

Notice that because $P_{-1}(\lambda) = 0$, it follows from Eq. (28) that $c_0 = 0$, and as a consequence, an even branch c_{2N} always lies lower than an odd branch $c_{2N\pm1}$, so that $c_{2N} < c_{2N\pm1}$. Then, we may conclude from Eq. (141) that

$$c_N = \frac{\mathcal{D}_N^+ - (-1)^N \mathcal{D}_N^-}{2}, \tag{147}$$

and, as a consequence,

$$\Gamma_{2\sigma+1}^{(N)} = \frac{1}{\pi c_N}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{t^{2\sigma+1}dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2}\left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}}\left[t^2 - (-1)^N \mathcal{D}_N^-\mathcal{D}_N^+\right]. \tag{148}$$

C. Soft Edges in the Two–Band Phase

To find the equations determining the end points $\mathcal{D}_N^{\pm}$ where the Dyson spectral density goes to zero, we start with the following formula from the theory of orthogonal polynomials [63]

$$n = 2c_n \int d\alpha(t)\frac{dV}{dt}P_n(t)P_{n-1}(t), \tag{149}$$

also known as a "string equation". Let us use expansion Eq. (136) to evaluate the integral entering Eq. (149) in the limit $n = N \gg 1$. It is easy to see that

$$N = 2c_N\sum_{k=1}^{p} d_k \int d\alpha(t) P_N(t) P_{N-1}(t) t^{2k-1} = 2c_N\sum_{k=1}^{p} d_k \Gamma_{2k-1}^{(N)}, \tag{150}$$

where $\Gamma_{2k-1}^{(N)}$ is given by Eq. (146). Then, we immediately obtain the relationship

$$N = \frac{2}{\pi}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2}\left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}}\frac{dV}{dt}\left[t^2 + c_N^2 - c_{N-1}^2\right]. \tag{151}$$

This result, rewritten for $n = N - 1$, yields in the large–N limit,

$$N = \frac{2}{\pi} \int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2} \left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}} \frac{dV}{dt} \left[t^2 + c_{N-1}^2 - c_N^2\right]. \tag{152}$$

Equations (151) and (152) bring us two integral equations whose solutions determine the end points $\mathcal{D}_N^{\pm}$,

$$\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{t^2 dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2} \left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}} \frac{dV}{dt} = \frac{\pi N}{2}, \tag{153}$$

and

$$\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dt}{\left[\left(\mathcal{D}_N^+\right)^2 - t^2\right]^{1/2} \left[t^2 - \left(\mathcal{D}_N^-\right)^2\right]^{1/2}} \frac{dV}{dt} = 0. \tag{154}$$

As $\mathcal{D}_N^- \to 0$, Eq. (153) coincides with the integral equation (66) for a single–band phase. In the same limit, Eq. (154) becomes equivalent to the assertion $\nu_D(0) = 0$, with ν_D being the spectral density in a single–band phase. This corresponds to the point of merging of two eigenvalue cuts.

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